A sufficient condition for the quasipotential to be the rate function of the invariant measure of countable-state mean-field interacting particle systems

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Joint work with Rajesh Sundaresan (Indian Institute of Science)

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- $\{(X_n^N(t), 1 \le n \le N), t \ge 0\}$ is a Markov process on $\mathcal{Z}^N$. $\{\mu^N(t), t \ge 0\}$ is a Markov process on $M_1(\mathcal{Z})$.

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- μ^N is a Markov process on $M_1(\mathcal{Z})$ with infinitesimal generator

$$L^N f(\xi) = \sum_{(z,z')\in\mathcal{E}} N\xi(z)\lambda_{z,z'}(\xi)\left[f\left(\xi + \frac{\delta_{z'}}{N} - \frac{\delta_z}{N}\right) - f(\xi)\right].$$

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- Under suitable conditions, L^N possesses a unique invariant probability measure $\wp^N$.
- Goal: study the large deviations of the family $\{\wp^N, N \geq 1\}$.

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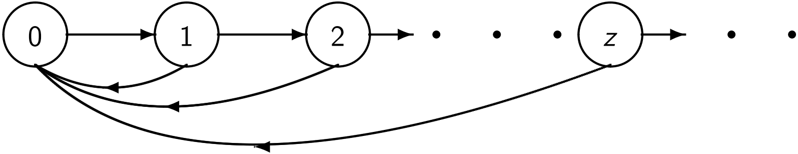

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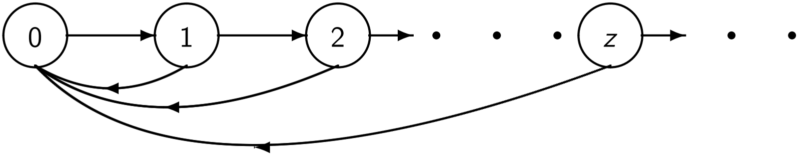

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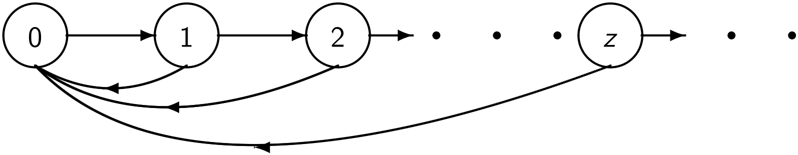

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$$\lambda_{z,0}(\xi) = c_z \exp\{-\langle c, \xi \rangle\},$$
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- Typical behavior of μ^N (mean-field limit): Let $\mu^N(0) \to \nu$ as $N \to \infty$. Assume that $\lambda_{z,z'}$ are Lipschitz continuous. Then $\{(\mu^N(t), 0 \le t \le T)\}$ converges in probability to the solution to the McKean-Vlasov equation:

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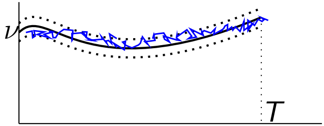
- Thus, μ^N is a small random perturbation of the above ODE.



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Process-level large deviations of μ^N

Theorem (Léonard (1995), Borkar and Sundaresan (2012))

Let $\nu_N \to \nu$ weakly. Then $\mu^N_{\nu_N}$ satisfies the LDP on $D([0,T], M_1(\mathcal{Z}))$ with rate function $S_{[0,T]}(\cdot|\nu)$ defined as follows. If $\mu_0 = \nu$ and $[0,T] \ni t \mapsto \mu_t \in M_1(\mathcal{Z})$ is absolutely continuous,

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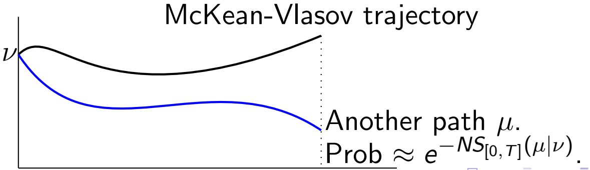

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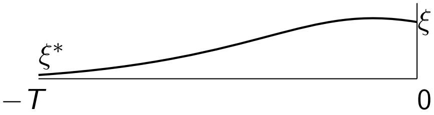

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- Small noise diffusions (Freidlin and Wentzell (1984)), finite-state mean-field models (Borkar and Sundaresan (2012)), reaction-diffusion equations (Sowers (1992), Cerrai and Röckner (2004)), stochastic wave equation (Martirosyan (2017)).

Large deviations of $\wp^N$: A counterxample

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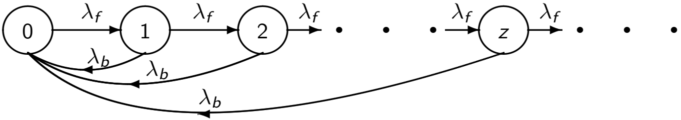

The stationary law of each particle is

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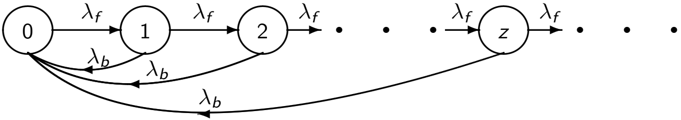

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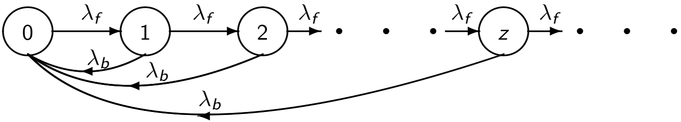

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- Let $\iota(z) = z$, $\vartheta(z) = z \log z$.
- If $\xi \in M_1(\mathcal{Z})$ is such that $\langle \xi, \iota \rangle < \infty$ and $\langle \xi, \vartheta \rangle = \infty$, then $V(\xi) = \infty$ but $H(\xi \| \xi^*) < \infty$.

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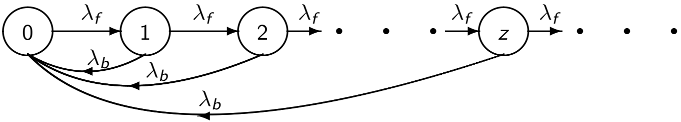

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- In particular, $V \neq H(\cdot\|\xi^*)$.

Assumptions and main result

- Assumptions:
  - There exist positive constants $\overline{\lambda}$ and $\underline{\lambda}$ such that

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Theorem

Under the above assumptions, the family $\{\wp^N, N \geq 1\}$ satisfies the LDP on $M_1(\mathcal{Z})$ with rate function V.

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  - The strong Markov property of μ^N.

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- for each $K \subset M_1(\mathcal{Z})$ compact and $s > 0$, $\mathcal{K} = \bigcup_{\nu \in K} \Phi_\nu(s)$ is a compact subset of $D([0, T], M_1(\mathcal{Z}))$;

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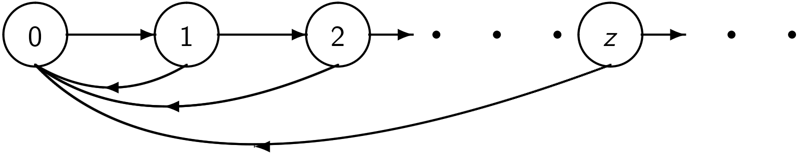
- We can show that $\{\mu_\nu^N\}$ satisfies the uniform LDP over the class of compact subsets of $M_1(\mathcal{Z})$.

Properties of the quasipotential

- Recall the transition graph:



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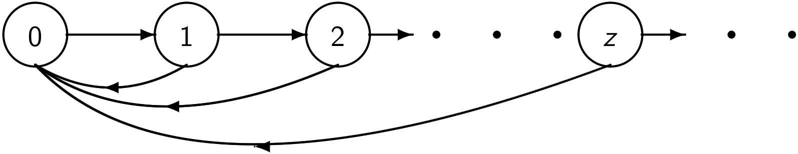

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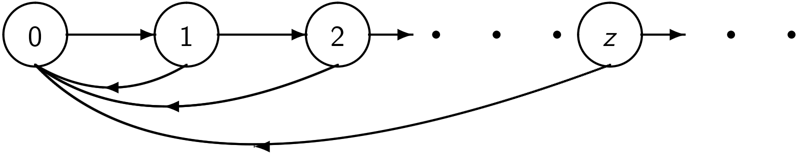

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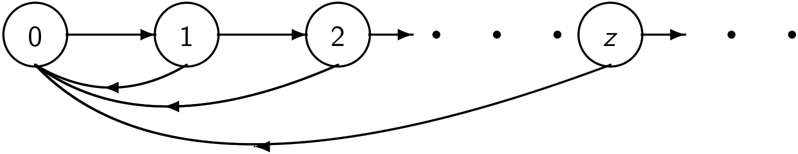

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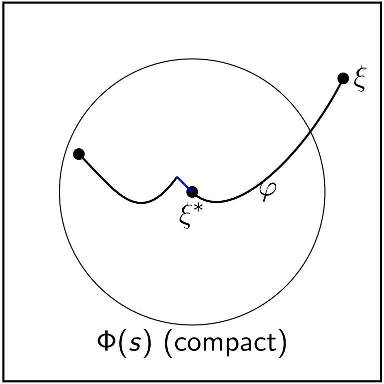
  - Construct a small-cost trajectory connecting ξ_n to ξ.

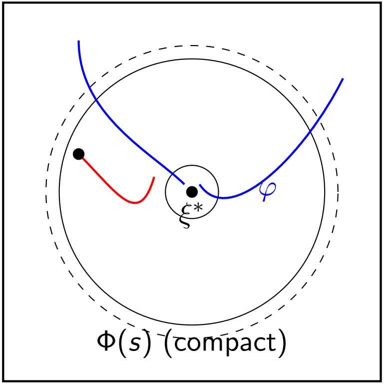
Proof sketch: Lower bound



- $\wp^N(\text{nbd}(\xi)) \geq \frac{1}{2} P(\mu^N_{\text{nbd}(\xi^*)} \in \text{nbd}(\varphi)) \geq \exp\{-N(V(\xi) + \gamma)\}$.
- The second inequality uses the uniform LDP over compact subsets of $M_1(\mathcal{Z})$.

Proof sketch: Upper bound

$$
\begin{aligned}
&\wp^N(\sim \mathrm{nbd}(\Phi(s))) \\
&\le \exp\{-Ns\} + P(\mu^N_{\Phi(s)}(T) \notin \mathrm{nbd}(\Phi(s))) \\
&\le \exp\{-Ns\} \\
&\quad + P(\mu^N_{\Phi(s)} \text{ does not hit } \mathrm{nbd}(\xi^*)) \\
&\quad + P(\mu^N_{\mathrm{nbd}(\xi^*)} \in \mathrm{nbd}(\varphi)) \\
&\le \exp\{-N(s-\gamma)\}
\end{aligned}
$$



- The first inequality uses exponential tightness.
- The second inequality uses the continuity of V under the convergence of ϑ-moments, and the strong Markov property.
- The third inequality uses the uniform LDP over compact subsets of $M_1(\mathcal{Z})$.

Summary and future directions

- Summary: LDP for the invariant measure in countable-state mean-field models.
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Reference: arXiv:2110.12640

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Thank you