A sufficient condition for the quasipotential to be the rate function of the invariant measure of countable-state mean-field interacting particle systems

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- Goal: study the large deviations of the family $\{\wp^N, N \geq 1\}$.

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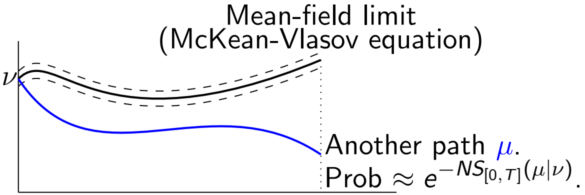

- Assume that ξ^* is the unique global attractor of the McKean-Vlasov equation.
- Let $\mu^N(0) \to \nu$ in $M_1(\mathcal{Z})$. Then the process $\{\mu^N\}$ satisfies the LDP on $D([0,T], M_1(\mathcal{Z}))$ with rate function $S_{[0,T]}(\cdot|\nu)$.

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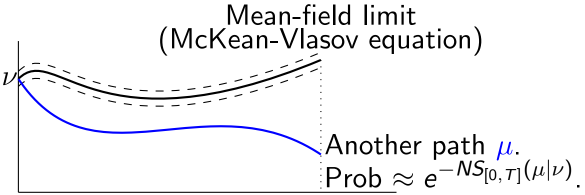

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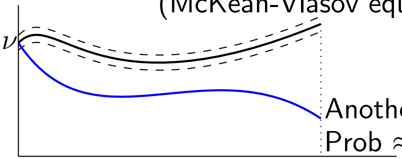

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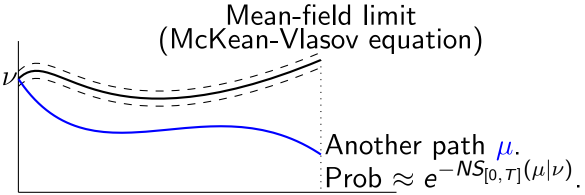

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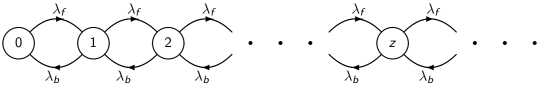
- V is a candidate rate function for the family $\{\wp^N\}$.
  - This is indeed the case in many models, e.g., finite-state mean-field models, small-noise diffusions.

A counterexample

- Consider N independent, identical, positive recurrent M/M/1 queues. Let ξ^* be the stationary distribution of one queue.



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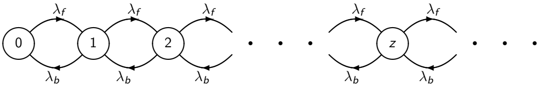

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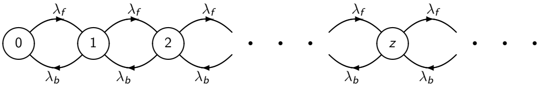

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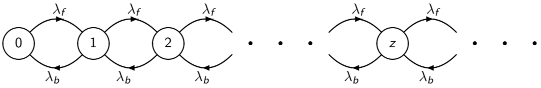

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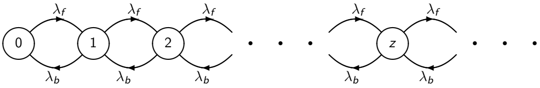

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  - If $\xi \in M_1(\mathcal{Z})$ is such that $\sum z\xi(z) < \infty$ and $\sum \vartheta(z)\xi(z) = \infty$, then $V(\xi) = \infty$ but $I(\xi\|\xi^*) < \infty$.

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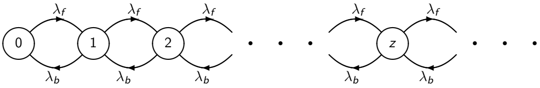

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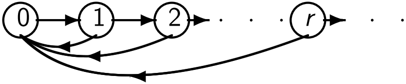
- Thus, the quasipotential V does not govern the LDP for the family $\{\wp^N\}$.

Assumptions and main result

- We make the following assumptions on the mean-field model.
  - Transition graph:



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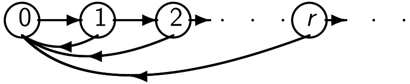

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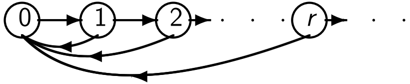

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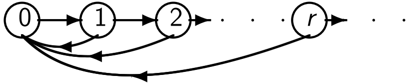

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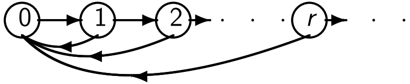

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- Under the above assumptions, we first show that, for each $N \geq 1$, there is a unique invariant measure $\wp^N$ for μ^N.

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  - The process-level uniform LDP for $\{\mu^N\}$ over compact subsets of $M_1(\mathcal{Z})$.
  - The strong Markov property of μ^N.
  - Exponential tightness of $\{\wp^N\}$: $\wp^N(\{\xi : \langle \xi, \vartheta \rangle \leq M\}^C) \leq \exp\{-NM'\}$ for all N.

Summary and future directions

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- Future directions:
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Reference: arXiv:2110.12640

Thank you