The Four Levels of Fixed-Points in Mean-Field Models

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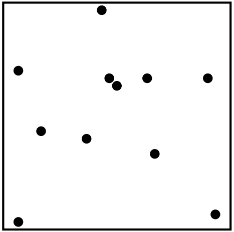
Introduction



- p is some property of a particle, e.g., collision probability of a node in a WiFi network.
- Attribute this property to each particle.
- Interaction among the particles create a mean-field.
- T is the response map to the induced mean-field.
- Self consistency demands that $T(p) = p$. That is, p is a fixed-point of T.
- Goal of this talk: explain the four levels of fixed-points in mean-field models and their connections.

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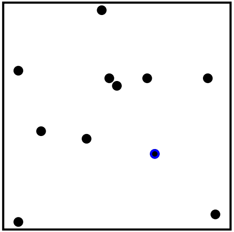

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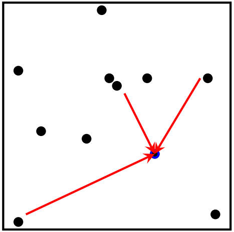

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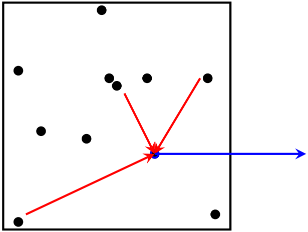

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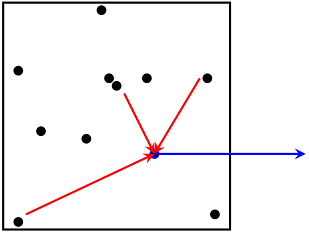

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- Markov evolution. A particle at time t makes a $z \to z'$ transition at rate $\lambda_{z,z'}(\mu_N(t))$.

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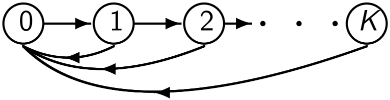

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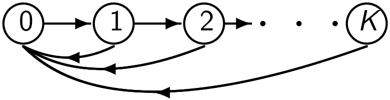

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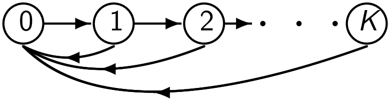

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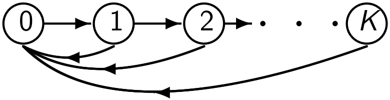

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- If c_i's are decreasing, then there is a unique fixed-point for G.

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- Analogy with thermodynamics: ideal gas law.

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- That is, $\dot{\xi}^*(t) = (\Lambda_{\xi^*(t)})^* \xi^*(t), \ t \geq 0$, the McKean-Vlasov equation (a non-linear ODE).

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- Conversely, any stationary Level-3 fixed point $\xi(\cdot)$ is a Level-2 fixed-point.

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- But the set $\mathcal{T}(\{R : R \circ \pi_t^{-1} = Q(t), t \geq 0\})$ can possibly contain other probability distributions.
- If the transition rates are Lipschitz continuous, then there is a one-one correspondence between Level-3 and Level-4. (non-linear martingale problem)

The four levels - Summary

Property	Space	Fixed-point equation
Macroscopic observables		$G(\gamma^*) = \gamma^*$
Dist. over states	$M_1(\mathcal{Z})$	$m(\xi^*) = \xi^*$
Evolution of dist.	$D([0,T], M_1(\mathcal{Z}))$	$\mathcal{M}(\xi^*(\cdot)) = \xi^*(\cdot)$
Dist. over trajectories	$M_1(D([0,T], \mathcal{Z}))$	$\mathcal{T}(Q^*) = Q^*$

Making the analysis rigorous

Mean-field convergence

- Let $\mu_N(0) \to \nu$ in distribution as $N \to \infty$ for some deterministic ν.
- Then the process $(\mu_N(t), t \in [0, T])$ converges to the solution to the McKean-Vlasov equation

$$\dot{\mu}(t) = \Lambda^*_{\mu(t)}\mu(t),\ \mu(0) = \nu,\ t \in [0, T], \tag{1}$$

in probability.

Making the analysis rigorous

Mean-field convergence

- Let $\mu_N(0) \to \nu$ in distribution as $N \to \infty$ for some deterministic ν.
- Then the process $(\mu_N(t), t \in [0, T])$ converges to the solution to the McKean-Vlasov equation

$$\dot{\mu}(t) = \Lambda^*_{\mu(t)}\mu(t),\ \mu(0) = \nu,\ t \in [0, T], \tag{1}$$

  in probability.

Decoupling approximation

- Consider two tagged particles, 1 and 2.
- Let the initial conditions be exchangeable and let $\mu_N(0) \to \nu$ in distribution as $N \to \infty$ for some deterministic ν.
- Then $(X_1^N(t), X_2^N(t))$ converges in distribution to $(Y_1(t), Y_2(t))$ where both $Y_1(t)$ and $Y_2(t)$ are distributed as the solution to (1) at time t with initial condition ν, and they are independent.

Beyond the fixed-point analysis

- Performance analysis when there is a unique fixed-point.
  - For the standard WiFi protocol, a fixed-point of $G(\gamma^*) = \gamma^*$ is a good approximation of the collision probability.

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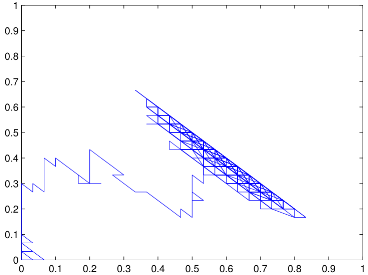

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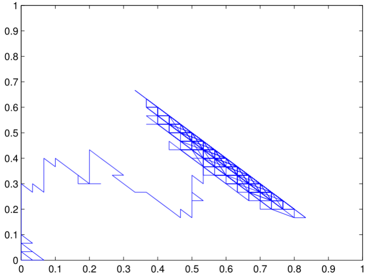

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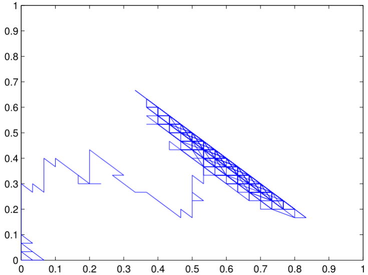

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Thank you