Large Time Behaviour and Metastability in Mean-Field Interacting Particle Systems

A Thesis
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Abstract

This thesis studies the large time behaviour and metastability in weakly interacting Markov processes with jumps. Our motivation is to quantify the large time behaviour of various networked systems that arise in practice.

The first set of results are for finite-state mean-field interacting particle systems. We first obtain a sharp estimate (in the exponential scale) on the time required for convergence of the empirical measure process of the N-particle system to its invariant measure; we show that when time is of the order of $\exp\{N\Lambda\}$ for a suitable constant $\Lambda > 0$, the process has mixed well and it is close to its invariant measure. We then obtain large-N asymptotics of the second largest eigenvalue of the generator associated with the empirical measure process when it is reversible with respect to its invariant measure. We show that its absolute value scales as $\exp\{-N\Lambda\}$. The main tools used in establishing these results are the large deviation properties of the empirical measure process from its large-N limit. As an application of the study of the large time behaviour, we also show the convergence of the empirical measure of the system of particles to a global minimum of a certain 'entropy' function when particles are added over time in a controlled fashion. The controlled addition of particles is analogous to the cooling schedule associated with the search for a global minimum of a function using the simulated annealing algorithm.

We then consider an extension of this finite-state mean-field model in which the particles are subject to a fast varying random environment. The second result of this thesis is the path-space large deviation principle (LDP) for the joint law of the empirical measure process of the particles and the occupation measure process of the fast environment. This extends previous results known for two time scale diffusions to two time scale mean-field models with jumps. Our proof is based on the method of stochastic exponentials. We characterise the rate function by studying a certain variational problem associated with an exponential martingale.

The third result is on the asymptotics of the invariant measure in countable-state mean-field models. The Freidlin-Wentzell quasipotential is the usual candidate rate function for the sequence of invariant measures indexed by the number of particles. We first provide two counterexamples where the quasipotential is not the rate function. The quasipotential arises from finite horizon considerations. However there are certain barriers that cannot be surmounted easily in any finite time horizon, but these barriers can be crossed in the stationary regime. Consequently, the quasipotential is infinite at some points where the rate function is finite. After highlighting this phenomenon, we study some sufficient conditions on a class of interacting particle systems under which one can continue to assert that the Freidlin-Wentzell quasipotential is indeed the rate function.

Keywords

Mathematics Subject Classifications (MSC 2020)

Primary	60F10	Large deviations
Secondary	60J74	Jump processes on discrete state spaces
	82C22	Interacting particle systems in time-dependent statistical mechanics
	60K35	Interacting random processes; statistical mechanics type models; percolation theory
	60K37	Processes in random environments
	47A75	Eigenvalue problems for linear operators
	68M20	Performance evaluation, queueing, and scheduling in the context of computer systems
	90B15	Stochastic network models in operations research

Keywords and phrases

Mean-field interaction, large deviations, metastability, exit from a domain, large time behaviour, second eigenvalue problem, simulated annealing, time scale separation, averaging principle, invariant measure, static large deviation, Freidlin-Wentzell quasipotential, relative entropy

Contents

Chapter 1

Introduction

This thesis studies the large time behaviour and metastability in Markovian mean-field interacting particle systems. Our motivation to study such questions comes from the performance analysis of networked system such as loss networks, wireless local area networks, load balancing networks, etc. Many of these networked systems can be abstracted using suitable models of Markovian mean-field interacting particle systems, and the goal of this thesis is to understand and quantify the large time behaviour of these models.

Let us begin with an illustration of the metastability phenomenon using a simple example of a double-well potential. Consider the one-dimensional dynamical system $dX_t = -U'(X_t)dt$, $t \geq 0$, where the function U is depicted in Figure 1.1(a). The function U has two local minima, labelled at points a and b, and a critical point at the origin. If this system is initiated at a point to the left of a (resp. to the right of b), it would move towards the local minimum a (resp. b), as shown by the black arrows. We say that a and b are *stable equilibria* of the dynamical system. If the system is started at the origin, it stays at the origin forever. We now consider a noisy perturbation of this dynamical system by adding a small amount of Gaussian noise. Let $B_t, t \geq 0$, denote a standard one-dimensional Brownian motion and consider the noisy system $dX_t^\varepsilon = -U'(X_t^\varepsilon)dt + \sqrt{\varepsilon}dB_t$, $t \geq 0$, where $\varepsilon > 0$ is a small parameter. Since the noise is small, similar to the deterministic dynamical system, trajectories starting at points to the left of a (resp. to the right b) move towards a (resp. b). However, since the system is noisy, trajectories that stay in a neighbourhood of b could potentially climb the barrier and move to a neighbourhood of a, and vice-versa. Such a transition from a neighbourhood of b to a neighbourhood of a is depicted in Figure 1.1(b). It turns out that such transitions can be observed over time durations of the form $\exp\{\text{constant}/\varepsilon\}$ for a suitable constant. Thus we find that the trajectories of the above noisy system exhibit different behaviour over different time scales. On one hand, if we focus on a fixed time duration and consider the system for a

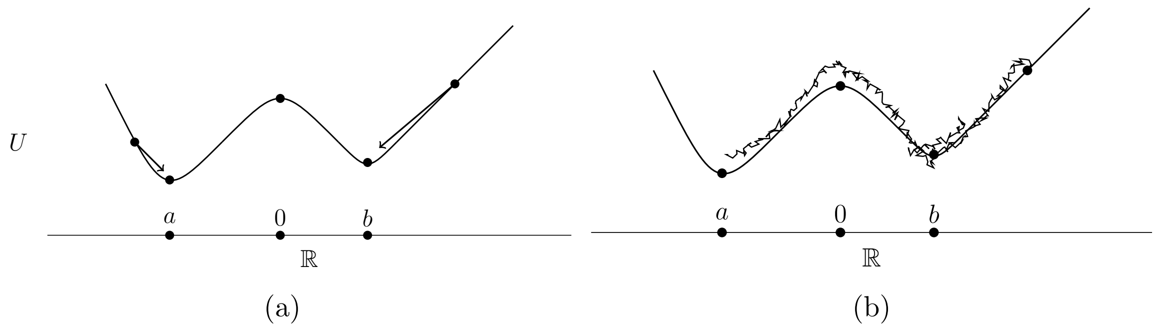

Figure 1.1: Illustration of the metastability phenomenon using the double-well potential. Figure (a) depicts trajectories of the dynamical system under consideration and figure (b) depicts a transition from a neighbourhood of b to a neighbourhood of a when this dynamical system is subject to a small random perturbation.

small $\varepsilon > 0$, then it would most likely track a trajectory of the deterministic dynamical system. On the other hand, if we consider time durations of the form $\exp\{\text{constant}/\varepsilon\}$ for small $\varepsilon > 0$, then we would observe transitions from one equilibrium to the other. Such a phenomenon where a system exhibits different dynamical behaviour over different time scales is referred to as metastability.

For the models of mean-field interacting particle systems considered in this thesis, the empirical measure of the system of particles can be viewed as a small random perturbation of a certain limiting dynamical system that evolves on the space of probability measures on a suitable set. The vector-field of this limiting dynamical system creates many stable equilibria, limit cycles, and/or chaotic attractors (see Figure 1.2 for an illustration). Typically a stable equilibrium is associated with a certain performance metric of the system under study; for instance, in the context of a wireless local area network, a stable equilibrium is associated with a certain average throughput of the system. Transitions between stable equilibria of the state space occur in these systems over large time durations, see Figure 1.2 for a transition from a neighbourhood of a to a neighbourhood of b. Therefore, from the point of view of performance analysis of these systems, one is interested in quantifying (i) the mean time spent by the system in a neighbourhood of an equilibrium, (ii) the probability of transiting to a neighbourhood of a given equilibrium before reaching another one, etc. Furthermore, owing to the presence of multiple equilibria, the system may get trapped in some undesired equilibrium for a long time. This results in slower convergence of the system to its stationary behaviour. Thus, one is also interested in quantifying the mixing time of these systems. The goal of this thesis is to quantify such large time behaviour in models of Markovian mean-field interacting particle systems.

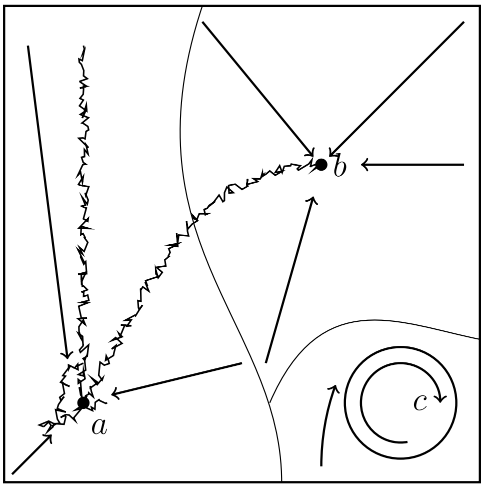

Figure 1.2: Illustration of the metastability phenomenon using an abstract dynamical system. The points a and b are two stable equilibria and c is a stable limit cycle of the dynamical system. Thick black arrows indicate the trajectories of the unperturbed dynamical system. Light black boundaries demarcate the basins of attraction of these ω-limit sets. The zig-zag curve indicates a sample path of a small random perturbation of the dynamical system over a large time duration. The random process starts in the basin of attraction of the equilibrium a, reaches a small neighbourhood of a, spends a lot of time there, and then makes a transition to a small neighbourhood of the equilibrium b.

1.1 Motivating examples

The motivation to study the questions addressed in this thesis is to quantify the large time behaviour and metastability phenomenon in a variety of applications that arise in practice. These applications include engineered system such as load balancing networks [2, 1, 65, 64, 41], wireless local area networks [10, 6, 14, 51, 75, 11], retrial queues [4], loss networks [46], and natural systems such as grammar acquisition, sexual evolution [68, 69], and epidemic spread [54, 3], to name a few. For a generic introduction to metastability and other stochastic models where it arises, see [67, 16]. In this section, we describe two applications that can be modelled using the mean-field interacting particle systems considered in this thesis. These two examples are (i) dynamic alternate routing in loss networks, and (ii) distributed medium access control protocols in wireless local area networks.

1.1.1 Loss networks with dynamic alternate routing

Consider N links (also called particles). Each link has C units of resources available with it. Calls arrive to each link according to independent Poisson point processes of rate λ. Each call requires one unit of resource. When a link has at most $C-1$ calls in progress, an incoming call to that link is accepted and it occupies a random amount of time on that link which is distributed according to an exponential random variable with unit mean. When an arriving call to a link finds that it is fully occupied (i.e., when there are C calls in progress), then two other links are randomly picked and the call is rerouted to both. If both these links are not fully occupied, then the call behaves as two independent calls and they occupy both those links for independent random amounts of time distributed as exponential random variables with unit mean. If not (i.e., when at least one of those two links are fully occupied), the call gets rejected from the system. This model arises in the context of loss networks with dynamic alternate routing (see [40]).

Let $X_n^N(t)$ denote the number of calls in progress on the nth link at time t. This is the state of the nth particle at time t. Let $\mu^N(t) := \frac{1}{N}\sum_{n=1}^N \delta_{X_n^N(t)}$ denote the empirical measure of the states of all the particles at time t; $\mu^N(t)(c)$, $0 \leq c \leq C$, denotes the fraction of links with c calls in progress at time t. We can write down the transition rates of each particle in terms of the empirical measure μ^N. Consider a tagged particle n. Let c denote its state at time t and let ξ denote the empirical measure of the system of particles at time t. Then the transition rates of the nth particle at time t are as follows:

$$c \to \begin{cases} c-1 & \text{at rate } c, \text{ if } c \geq 1, \\ c+1 & \text{at rate } \lambda + N\lambda\xi(C) \times \frac{2(1-\xi(C))}{N-1}, \text{ if } c \leq C-1. \end{cases}$$

The first transition corresponds to a call leaving the nth link after completion and the second transition corresponds to an incoming call to the nth link. An increment in the number of calls on the nth link could either be because of the incoming request to the nth link (at rate λ) or because of a rerouted request from another link that is fully occupied (at rate $N\lambda\xi(C) \times \frac{2(1-\xi(C))}{N-1}$). We thus find that the transition rates of a particle depend on the states of the other particles only through the empirical measure of the states of all the particles. This model falls within the framework of a finite-state mean-field interacting particle system studied in Chapter 2.

For each $T > 0$, the empirical measure process $\{\mu^N(t), t \in [0,T]\}$ can be viewed as a small random perturbation of a certain dynamical system that evolves on the space of probability measure on $\{0, 1, \dots, C\}$. It can be shown that the noisy process converges to this limiting dynamical system as $N \to \infty$ in the space of trajectories, in probability. It is known that such models exhibit metastability [40]. For certain choices of the model parameters, it turns out that this dynamical system possesses two stable equilibria and an unstable equilibrium. Therefore, one can observe transitions (similar to the one depicted in Figure 1.1) between the stable equilibria of this dynamical system over large time durations. See Tibi [85] for some estimates on the exit times. More recently, Olesker-Taylor [66] established that under the regime where the system exhibits metastability, the mixing time of the process $\{\mu^N(t), t \geq 0\}$ is exponential in the number of links. In Chapter 2 of this thesis, we consider a generic finite-state mean-field interacting particle system and study its large time behaviour and convergence to stationary. From these results, we can quantify the large time behaviour and metastability in loss networks with dynamic alternate routing.

1.1.2 Wireless local area networks with multiple classes of users

Let there be N nodes in a wireless local area network (WLAN). Time is divided into slots. Each node has a state associated with it, which represents the probability of attempting a packet transmission in a slot. Since the network could be spread over a large geographical area, the nodes are grouped into C classes; every node that belongs to a class can hear the transmissions of every other node in that class. Figure 1.3 depicts an example network with 7 nodes and 3 classes. The interaction among the nodes comes from the distributed channel access algorithm executed by the nodes. This interaction results in the evolution of the state of each node in the following fashion: a node that incurs a collision upon a packet transmission moves to a different state with a reduced probability of attempt, and upon a successful transmission moves to another state with an increased probability of attempt. Figure 1.4 depicts the set of allowed transitions of a node; in typical WLAN implementations, the most aggressive state is 0 and the least aggressive state is K. A node moves from state i to state $i+1$ when it incurs a collision, and moves from state i to state 0 when a packet is successfully transmitted. Since multiple nodes could transmit at the same slot, the channel corresponding to a class of nodes could be in three different states in a given time slot: (i) an idle slot (denoted by state 0), (ii) a collision (state 2) or (iii) a successful packet transmission (state 1). We refer to the channel state corresponding to each class of nodes as the environment, i.e., at each time slot, the environment is an element of $\{0, 1, 2\}^C$ with the cth coordinate representing the channel state of the cth class of nodes. We now see how to translate this to an approximate continuous-time model for large N.

To describe the transition rates of the continuous time model, we shall consider a scaled version of the above discrete time model where each time slot is of duration $1/N$. Let p_i/N denote the attempt probability of a node in state i. Let A denote the interference matrix among the classes, specifically, $A_{c,d} = 1$ implies that a class c node's transmission is interfered by a class d node's transmission. Let $V_c = \{d : A_{c,d} = 1\}$ denote the classes that interfere with class

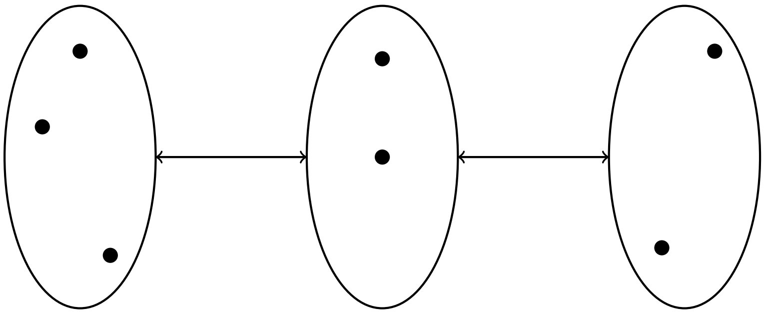

Figure 1.3: A wireless local area network with 3 classes and 7 users; interference among classes are indicated by arrows.

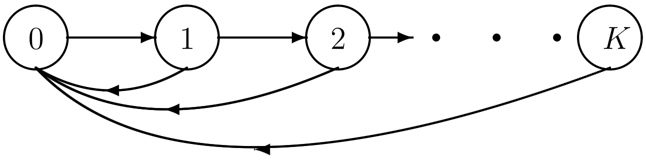

Figure 1.4: Set of allowed transitions for a particle in a WLAN.

c nodes' transmissions; in particular, $c \in V_c$. Consider an epoch. For each $i \in \{0, 1, \ldots, K\}$ and $c \in \{1, 2, \ldots, C\}$, let ξ_i^c denote the fraction of nodes (among the nodes in class c) in state i. Let $y \in \{0, 1, 2\}^C$ denote the state of the environment. At this epoch, a tagged node n_0 in state $i \neq 0$ in class c moves to state 0 when it successfully transmits a packet. This occurs when the following are true: (i) the tagged node transmits, (ii) there is no interference from the other nodes in class c (which occurs when every other node in class c could potentially transmit, i.e., there is no interference from any node in the set of classes V_c, but no one transmits), and (iii) there is no interference from the nodes in the other classes in V_c except c. The product of the probabilities of these events can be written as

$$\overbrace{\frac{p_i}{N}}^{\text{the tagged node } n_0 \text{ in state } i \text{ transmits}} \cdot \overbrace{\left(\prod_{d \in V_c} \mathbf{1}_{\{y_d = 0\}}\right) \cdot \left(\prod_{n \in c, n \neq n_0} \left(1 - \frac{p_{\text{state}(n)}}{N}\right)\right)}^{\text{all classes in } V_c \text{ are idle and no other node in class } c \text{ transmits}}$$
$$\times \prod_{d \in V_c, d \neq c} \left\{ \underbrace{\left(\prod_{d' \in V_d} \mathbf{1}_{\{y_{d'} = 0\}}\right) \cdot \left(\prod_{n \in d} \left(1 - \frac{p_{\text{state}(n)}}{N}\right)\right)}_{\text{all classes in } V_d \text{ are idle and no node in class } d \text{ transmits}} + \underbrace{\left(1 - \prod_{d' \in V_d} \mathbf{1}_{\{y_{d'} = 0\}}\right)}_{\text{some class in } V_d \text{ is not idle}} \right\},$$

where $\mathbf{1}_{\{.\}}$ denotes the indicator function and state(n) denotes the state of the nth node at the epoch under consideration. Scaling the above by N, and noting that $\prod_{n \in d}(1 - p_{\text{state}(n)}/N) \sim \exp\{-\sum_{i=0}^K p_i \xi_i^d\}$ and the term $\exp\{-\sum_{i=0}^K p_i \xi_i^c\}$ arising from the first line above can be absorbed in the product in the second line, the corresponding transition rate of the continuous time model at this epoch can be approximated as

$$p_i \left(\prod_{d \in V_c} \mathbf{1}_{\{y_d = 0\}} \right) \times \prod_{d \in V_c} \left\{ \left(\prod_{d' \in V_d} \mathbf{1}_{\{y_{d'} = 0\}} \right) \left(\exp\left\{ - \sum_{i=0}^{K} p_i \xi_i^d \right\} - 1 \right) + 1 \right\};$$

Similarly, a tagged node in state i moves to state $i+1$ when it incurs a collision. This occurs when (i) the tagged node transmits, and (ii) at least another node from either class c or any other class from V_c transmits. Proceeding as above, the transition rate of a class c node from state i to state $i+1$ is

$$p_i \left(\prod_{d \in V_c} \mathbf{1}_{\{y_d = 0\}} \right) \times \left[1 - \prod_{d \in V_c} \left\{ \left(\prod_{d' \in V_d} \mathbf{1}_{\{y_{d'} = 0\}} \right) \left(\exp\left\{ - \sum_{i=0}^{K} p_i \xi_i^d \right\} - 1 \right) + 1 \right\} \right].$$

We can also write down the transition rates of the environment. For example, a transition from the all-0 state to the state y with $y_c = 1$ and $y_d = 0$ for all $d \neq c$ (which happens when any node in class c makes a successful transmission) occurs with probability

$$\sum_{n \in c} \overbrace{\frac{p_{\text{state}(n)}}{N}}^{\text{node } n \text{ in class } c \text{ transmits}} \times \overbrace{\prod_{n' \in c, n' \neq n} \left(1 - \frac{p_{\text{state}(n')}}{N} \right)}^{\text{no other node in class } c \text{ transmits}}$$

As before, scaling the above with N, the corresponding transition rate for the continuous time model is

$$\left(N \sum_{i=0}^{K} p_i \xi_i^c \right) \times \exp\left\{ - \sum_{i=0}^{K} p_i \xi_i^c \right\}.$$

From the above description, we see that the transition rates of a node at time t depend on the state of the environment at time t and the empirical measure of the states (via the attempt probabilities) of all the nodes of its neighbouring classes at time t. We also see that the transition rates of the environment depend on the states of the nodes in that class, but only through their empirical measures. Further, the environment makes $O(N)$ many transitions over a given $O(1)$ time duration. Such a mean-field model where there is a time scale separation between the particles and the environment falls in to the framework of two time scale mean-field models studied in Chapter 3. In particular, if all the nodes are visible to each other, then one

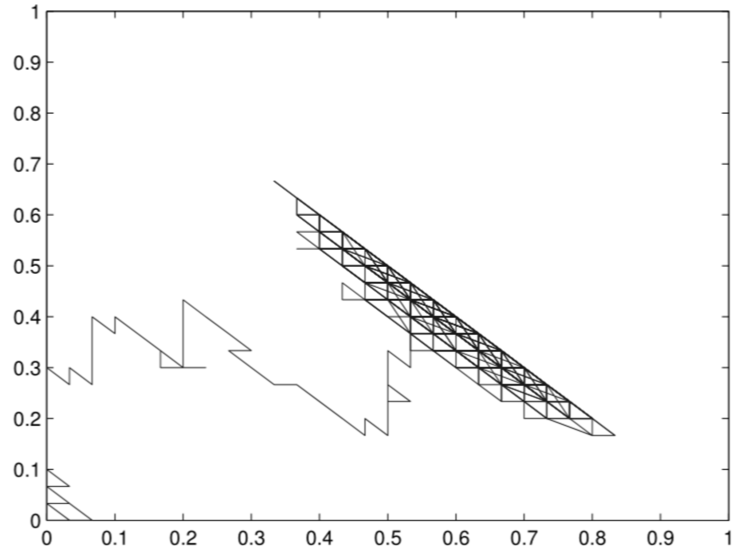

Figure 1.5: A sample path of the evolution of the empirical measure in a WLAN under the MAC protocol. The abscissa and the ordinate represent the fraction of nodes in states 0 and 1 respectively; the fraction of nodes in state 2 is such that the sum of the three is 1. The process starts near the point $(0.3, 0.7)$, spends a lot of time in a neighbourhood of $(0.6, 0.4)$, and then transitions to a neighbourhood of $(0, 0)$.

can write down the transition rates of a node in terms of the empirical measure of the states of all the nodes (without involving the channel state), which falls under the finite-state mean-field model considered in Chapter 2.

A study of the law of large numbers for the above two time scale model in the large-N regime has been done by Bordenave et al. [14] towards understanding the average throughput obtained by a node in a given class. The results of Chapter 3 of this thesis provide a finer asymptotic analysis, in the realm of large deviations, which enables us to study the large time behaviour and metastability in such systems.

We now demonstrate the metastability phenomenon in a WLAN with a single class of users using a numerical example. Let us consider a WLAN consisting of 30 nodes, accessing a common wireless medium using the standard 802.11 medium access control (MAC) protocol [52, Chapter 7]. In this example, each node can be in three states (i.e., $K = 2$). A sample path of the evolution of the fraction of nodes in each state is shown in Figure 1.5. This example is designed in such a way that there are two stable equilibria in the system, one near $(0.6, 0.4)$ which is the "good" equilibrium in the sense that every node gets a fair amount of access to the channel, and the other near $(0, 0)$, which is the "bad" equilibrium, where every node keeps on attempting for a transmission which results in nobody getting access to the channel. As shown in the figure, the system is initialised at a point near $(0.3, 0.7)$ and it remains in the neighbourhood of the "good" equilibrium for a long time (about 10^6 slots), but eventually it finds its way to the "bad" equilibrium $(0, 0)$. Thus, even though the desired operating point is near $(0.6, 0.4)$, the network eventually transits to a neighbourhood of $(0, 0)$ leading to a low network throughput. If we wait long enough, we would also see another transition from a neighbourhood of the "bad" equilibrium to the "good" equilibrium. This is an example of the metastability phenomenon observed in WLANs. The goal of this thesis is to quantify such phenomena.

1.2 Summary of results

Since this thesis studies the large time behaviour and metastability in mean-field models, as illustrated in the previous section, our primarily focus is to understand and quantify various rare events associated with mean-field interacting particle systems. In this thesis, we use the theory of large deviations to quantify the probabilities of rare events. Roughly speaking, the theory of large deviations quantifies the probabilities of rare events in the following form: for a sequence of $\mathcal{S}$-valued random variables $\{X^N, N \geq 1\}$ and a set $A \subset \mathcal{S}$, the probability of the event $\{X^N \in A\}$ is given by $\exp\{-N \inf_{x \in A} I(x)\}$, where $I : \mathcal{S} \to [0, \infty]$ is called the rate function. A more precise definition is given below.

Definition 1.1 (Large deviation principle)**.** Let $(\mathcal{S}, d_0)$ be a metric space. We say that a family $\{X^N, N \geq 1\}$ of $\mathcal{S}$-valued random variables defined on a probability space $(\Omega, \mathcal{F}, P)$ satisfies the large deviation principle (LDP) with rate function $I : \mathcal{S} \to [0, \infty]$ if

- (Compactness of level sets). For any $s \geq 0$, $\Phi(s) := \{x \in \mathcal{S} : I(x) \leq s\}$ is a compact subset of $\mathcal{S}$;
- (LDP lower bound). For any $\gamma > 0$, $\delta > 0$, and $x \in \mathcal{S}$, there exists $N_0 \geq 1$ such that

$$P(d_0(X^N, x) < \delta) \geq \exp\{-N(I(x) + \gamma)\}$$

  for any $N \geq N_0$;
- (LDP upper bound). For any $\gamma > 0$, $\delta > 0$, and $s > 0$, there exists $N_0 \geq 1$ such that

$$P(d_0(X^N, \Phi(s)) \geq \delta) \leq \exp\{-N(s - \gamma)\}$$

  for any $N \geq N_0$.

For the problems studied in this thesis, the metric space in the above definition may either be the space of probability measure-valued trajectories on a finite time interval or the space of probability measures on a suitable set. The first case corresponds to the study of the process-level large deviations of random processes arising in various contexts and the second corresponds to the study of the large deviations of the family of invariant measures of these processes. With this brief introduction to large deviations, we now summarise the results of this thesis.

Large time behaviour of finite-state mean-field models (Chapter 2): This chapter studies the large time behaviour and metastability in finite-state mean-field models. We consider N particles. Each particle has a state associated with it which comes from a finite set $\mathcal{Z}$. The state of the nth particle at time t is denoted by $X_n^N(t) \in \mathcal{Z}$. These states evolve over time in a Markovian fashion. The set of allowed transitions for the particles is described by a directed graph $(\mathcal{Z}, \mathcal{E})$. The empirical measure of the system of particles at time t is defined by $\mu^N(t) \coloneqq \frac{1}{N}\sum_{n=1}^N \delta_{X_n^N(t)}$, where δ denotes the Dirac measure on $\mathcal{Z}$. The states of the particles change over time as follows. At time t, a particle in state z makes a $z \to z'$ transition at rate $\lambda_{z,z'}(\mu^N(t))$, where $\lambda_{z,z'}$, $(z,z') \in \mathcal{E}$, are given functions on the space of probability measure on $\mathcal{Z}$. That is, the evolution of the state of a particle depends on the states of the other particles only through the empirical measure of the states of all the particles.

In this setting, under suitable assumptions on the model, we establish three results. The first result quantifies the convergence of the process μ^N to its invariant measure. We show that there is a constant Λ (which is described in terms of the structure of a certain limiting dynamical system) such that when time is of the order $\exp\{N(\Lambda+\delta)\}$ for any $\delta > 0$, the process μ^N has mixed well and it is close to its invariant measure, regardless of the initial condition (see Theorem 2.1). The main ingredients in the proof of this result are the quantifications of the large time behaviour of the process μ^N, which are of independent interest. We also prove that this constant Λ is sharp (see Theorem 2.2). The second result is on the asymptotics of the second largest eigenvalue of the generator of the process μ^N when it is reversible with respect to its invariant measure; we show that it scales as $\exp\{-N\Lambda\}$ (see Theorem 2.3). In the third result, we show that we can steer the process μ^N via controlled addition of the particles over time so that, with probability $1 - o(1)$ (as time becomes large), the process μ^N converges to a small neighbourhood of a global minimum of a certain entropy function (see Theorem 2.4). This is reminiscent of the simulated annealing algorithm to find a global minimum of a given function using noisy dynamical systems.

Process-level large deviations of two time scale mean-field models (Chapter 3): This chapter studies the large deviations of a two time scale mean-field interacting particle system.

This model is an extension of the model considered in Chapter 2. We consider N particles and an environment. Their states evolve over time in a Markovian fashion. Let $X_n^N(t)$ denote the state of the nth particle and $Y^N(t)$ denote the state of the environment at time t. The states of the particles come from a finite set $\mathcal{X}$ and that of the environment comes from a finite set $\mathcal{Y}$. As before, we define the empirical measure of the system of particles at time t as $\mu^N(t) := \frac{1}{N}\sum_{n=1}^N \delta_{X_n^N(t)}$. At time t, a particle in state x makes an $x \to x'$ transition at rate $\lambda_{x,x'}(\mu^N(t), Y^N(t))$, and the environment in state y makes a $y \to y'$ transition at rate $N\gamma_{y,y'}(\mu^N(t))$. That is, the transition rates of a particle depend on the states of the other particles through the empirical measure of the states of all the particles and the state of the environment, and the transition rates of the environment depend on the empirical measure. We also note that the environment evolves in a faster time scale; hence there is time scale separation between the particles and the environment.

In Chapter 3, under suitable assumptions on the above model, we prove a large deviation principle for the joint law of the empirical measure process of the particles and the occupation measure process of the fast environment (see Theorem 3.1). The rate function for this large deviation principle is governed by "costs" associated with trajectories on suitable path-spaces. Using this result and the results on the large time behaviour of finite-state mean-field models established in Chapter 2, we can study the large time behaviour and metastability in mean-field models with time scale separation.

Large deviations of the invariant measure in countable-state mean-field models (Chapter 4): This chapter studies mean-field interacting particle systems with a countable state space. Let $\mathcal{Z}$ denote the set of nonnegative integers and let $(\mathcal{Z}, \mathcal{E})$ denote a directed graph. As before, we consider N particles. Let $X_n^N(t)$ denote the state of the nth particle at time t. The empirical measure of the system of particles at time t is defined by $\mu^N(t) := \frac{1}{N}\sum_{n=1}^N \delta_{X_n^N(t)}$. At time t, a particle from state z moves to state z' at rate $\lambda_{z,z'}(\mu^N(t))$. Under suitable assumptions on the model, the Markov process μ^N possesses a unique invariant probability measure $\wp^N$, which is a probability measure on the space of probability measures on $\mathcal{Z}$. We study the large deviations of the family $\{\wp^N, N \geq 1\}$ in Chapter 4.

For a broad class of Markov processes such as small-noise diffusions, finite-state mean-field models, simple exclusion processes, etc., it is well-known that the Freidlin-Wentzell quasipotential is the rate function that governs the family of invariant measures. We first provide two counterexamples of countable-state mean-field models where the family of invariant measures satisfies the LDP whose rate function is governed by a certain relative entropy, which is not the same as the quasipotential. Specifically, we show that there are points in the state space where the rate function is finite, but the quasipotential is infinite. We then impose some assumptions on the model, restricting attention to situations where such issues do not arise, and show that the family of the invariant measures satisfies the large deviation principle on the space of probability measure on $\mathcal{Z}$ whose rate function is indeed governed by the quasipotential (see Theorem 4.1).

1.3 Organisation

This thesis is organised as follows. Chapter 2 studies the large time behaviour and metastability in finite-state mean-field models. Chapter 3 studies the process-level large deviations of finite-state mean-field models with time scale separation. Chapter 4 studies the large deviations of the invariant measure in countable-state mean-field models. In the beginning of each chapter, we describe the setting of the problem, notations, main results and novelties, and connections to the existing literature for the problems studied in that chapter. Chapter 5 concludes the thesis and discusses some open questions.

Chapter 2

Large Time Behaviour of Finite-State Mean-Field Models

2.1 The setting and main results

2.1.1 The setting

Let there be N particles. Each particle has a state associated with it which comes from a finite set $\mathcal{Z}$; the state of the nth particle at time t is denoted by $X_n^N(t) \in \mathcal{Z}$. The empirical measure of the system of particles at time t is defined by

$$\mu^N(t) \coloneqq \frac{1}{N} \sum_{n=1}^{N} \delta_{X_n^N(t)} \in \mathcal{M}_1(\mathcal{Z}),$$

where $\delta_\cdot$ denotes the Dirac measure on $\mathcal{Z}$. Here, $\mathcal{M}_1(\mathcal{Z})$ denotes the space of probability measures on $\mathcal{Z}$ equipped with a metric that generates the topology of weak convergence[1] on $\mathcal{M}_1(\mathcal{Z})$. Each particle has a set of allowed transitions; to define this, let $(\mathcal{Z}, \mathcal{E})$ be a directed graph with the interpretation that whenever $(z, z') \in \mathcal{E}$, a particle in state z is allowed to move from z to z'. To specify the interaction among the particles and the evolution of the states of the particles over time, for each $(z, z') \in \mathcal{E}$, we are given a function $\lambda_{z,z'} : \mathcal{M}_1(\mathcal{Z}) \to [0, \infty)$. We consider the generator Ψ^N acting on functions f on $\mathcal{Z}^N$ by

[1] Since $\mathcal{Z}$ is a finite set, the total variation metric on $\mathcal{M}_1(\mathcal{Z})$ generates this topology.

$$\Psi^N f(\mathbf{z}^N) = \sum_{n=1}^{N} \sum_{z'_n:(z_n,z'_n)\in\mathcal{E}} \lambda_{z_n,z'_n}(\overline{\mathbf{z}^N})(f(\mathbf{z}^N_{n,z_n,z'_n}) - f(\mathbf{z}^N));$$

here $\overline{\mathbf{z}^N} = \frac{1}{N}\sum_{n=1}^N \delta_{z_n} \in \mathcal{M}_1(\mathcal{Z})$ denotes the empirical measure associated with the configuration $\mathbf{z}^N \in \mathcal{Z}^N$, and $\mathbf{z}^N_{n,z_n,z'_n}$ denotes the resultant configuration of the particles when the nth particle changes its state from z_n to z'_n.

We make the following assumptions on the model:

(A1) The graph $(\mathcal{Z},\mathcal{E})$ is irreducible.

(A2) The functions $\lambda_{z,z'}(\cdot)$, $(z,z') \in \mathcal{E}$, are Lipschitz continuous on $\mathcal{M}_1(\mathcal{Z})$ and there exist positive constants c, C such that $c \le \lambda_{z,z'}(\xi) \le C$ for all $(z,z') \in \mathcal{E}$ and all $\xi \in \mathcal{M}_1(\mathcal{Z})$.

Let $D([0,\infty),\mathcal{Z}^N)$ denote the space of $\mathcal{Z}^N$-valued functions on $[0,\infty)$ that are right continuous with left limits (càdlàg), equipped with the Skorohod-J_1 topology (see [34, Chapter 3]). Since the transition rates are bounded (by assumption (A2)), the $D([0,\infty),\mathcal{Z}^N)$-valued martingale problem for Ψ^N is well posed (see [34, Exercise 15, Section 4.1]); therefore, given an initial configuration of the particles $(X^N_n(0), 1 \le n \le N) \in \mathcal{Z}^N$, we have a Markov process $\big((X^N_n(t), 1 \le n \le N), t \ge 0\big)$ whose sample paths are elements of $D([0,\infty),\mathcal{Z}^N)$. To describe the process in words, a particle in state z at time t moves to state z' at rate $\lambda_{z,z'}(\mu^N(t))$ independent of everything else; i.e., the evolution of the state of a particle depends on the states of the other particles via the empirical measure of the states of all the particles, hence the name mean-field interaction. Note that the empirical measure process $(\mu^N(t), t \ge 0)$ is also a Markov process with state space $\mathcal{M}^N_1(\mathcal{Z})$ which is the set of elements of $\mathcal{M}_1(\mathcal{Z})$ that can arise as empirical measures of N-particle configurations on $\mathcal{Z}^N$. Its generator L^N acting on functions f on $\mathcal{M}^N_1(\mathcal{Z})$ is given by

$$L^N f(\xi) = N \sum_{(z,z')\in\mathcal{E}} \xi(z)\lambda_{z,z'}(\xi)\left[f\left(\xi + \frac{\delta_{z'}}{N} - \frac{\delta_z}{N}\right) - f(\xi)\right].$$

Since μ^N is a Markov process on a finite state space, and since the graph $(\mathcal{Z},\mathcal{E})$ of the allowed particle transitions is irreducible (Assumption (A1)), there exists a unique invariant probability measure for μ^N, which we denote by $\wp^N$. Also, let P_ν denote the law of $(\mu^N(t), t \ge 0)$ with initial condition $\mu^N(0) = \nu \in \mathcal{M}^N_1(\mathcal{Z})$ (i.e. the solution to the $D([0,\infty),\mathcal{M}_1(\mathcal{Z}))$-valued martingale problem for L^N with initial condition $\nu \in \mathcal{M}^N_1(\mathcal{Z})$) and let E_ν denote integration with respect to P_ν; in both P_ν and E_ν we suppress the dependence on N for ease of readability.

2.1.2 Main results

Let us now discuss the main results of this chapter. All results of this chapter are established under assumptions (A1) and (A2) on the particle system, and a further assumption (B1) on the structure of the large time behaviour of the ODE (2.1) (see Section 2.3).

2.1.2.1 Convergence to the invariant measure

Our first main result is on the time required for the process μ^N to equilibrate. This time grows at an exponential rate with the number of particles N where the rate is the constant $\Lambda > 0$ which will be defined in (2.9).

Theorem 2.1. *Given $\delta > 0$ there exist $\varepsilon > 0$ and $N_0 \geq 1$ such that, with $T = \exp\{N(\Lambda + \delta)\}$,*

$$\sup_{\nu \in \mathcal{M}_1^N(\mathcal{Z})} \left| E_\nu(f(\mu^N(T))) - \langle f, \wp^N \rangle \right| \leq \|f\|_\infty \exp\{-\exp(N\varepsilon)\}$$

for all $N \geq N_0$ and all bounded Borel-measurable functions f on $\mathcal{M}_1(\mathcal{Z})$.

The result says that when time is of the order $\exp\{N(\Lambda + \delta)\}$ for any $\delta > 0$, the process has mixed well and it is close to its invariant measure. The proof of this result is based on the study of the large time behaviour of the process μ^N. Before we describe this, let us mention a well-known law of large numbers for the process μ^N [59, 39, 83, 6]. This will not only pave the way for a suitable description of the constant Λ but also lead us to a converse of Theorem 2.1 and the significance of Λ.

Assume (A1) and (A2), and suppose that the initial conditions $\{\mu^N(0), N \geq 1\}$ converge weakly to a deterministic measure $\nu \in \mathcal{M}_1(\mathcal{Z})$ as $N \to \infty$. Then for any fixed $T > 0$, the empirical measure process $\{(\mu^N(t), t \in [0, T]), N \geq 1\}$ converges in $D([0, T], \mathcal{M}_1(\mathcal{Z}))$, in probability, to the solution to the ODE

$$\dot{\mu}(t) = \Lambda^*_{\mu(t)} \mu(t),\ t \in [0, T],\ \mu(0) = \nu, \tag{2.1}$$

where, for any $\xi \in \mathcal{M}_1(\mathcal{Z})$, Λ_ξ denotes the $|\mathcal{Z}| \times |\mathcal{Z}|$ rate matrix[1] when the empirical measure is ξ, Λ^*_ξ denotes its transpose, $D([0, T], \mathcal{M}_1(\mathcal{Z}))$ denotes the space of $\mathcal{M}_1(\mathcal{Z})$-valued càdlàg functions on $[0, T]$ equipped with the Skorohod-J_1 topology (we assume that all paths are left continuous at T), and both $\mu(t)$ and $\dot{\mu}(t)$ are viewed as column vectors. The above ODE is

[1] The rate matrix is given by $\Lambda_\xi(z, z') = \lambda_{z,z'}(\xi)$ when $(z, z') \in \mathcal{E}$, $\Lambda_\xi(z, z') = 0$ when $(z, z') \notin \mathcal{E}$, and $\Lambda_\xi(z, z) = -\sum_{z' \neq z} \lambda_{z,z'}(\xi)$ for all $z \in \mathcal{Z}$.

referred to as the McKean-Vlasov equation. The above convergence result enables one to view the process μ^N as a small random perturbation of the ODE (2.1).

We now elaborate on the large time behaviour of μ^N. Suppose that the limiting McKean-Vlasov equation (2.1) has multiple ω-limit sets (multiple stable equilibria and/or limit cycles). If we focus on a fixed time interval $[0, T]$, let the number of particles $N \to \infty$, and let the initial conditions $\mu^N(0)$ converge weakly to a deterministic limit ν, then the mean-field convergence suggests that the empirical measure process tracks the solution to the McKean-Vlasov equation (2.1) over $[0, T]$ starting at ν. If we then let $T \to \infty$, the solution to the McKean-Vlasov equation goes to an ω-limit set of (2.1) depending on the initial condition ν. On the other hand, for a large but fixed N, the process would track the McKean-Vlasov equation with high probability and, as time becomes large, would thus enter a neighbourhood of the ω-limit set corresponding to the initial condition ν; however, because of the randomness in the finite-N system, the process can exit the basin of attraction of this ω-limit set. It is then likely to remain in a neighbourhood of another ω-limit set for a large amount of time before transiting to the next one, and so on. These are examples of *metastable phenomena,* and it turns out that the sojourn times in the basin of attraction of an ω-limit set are of the order $\exp\{O(N)\}$, as we shall soon see. The proof of Theorem 2.1 exploits quantitative estimates of the following metastable phenomena,

(i) the mean time spent by the process near an ω-limit set,

(ii) the probability of first reaching a particular ω-limit set's neighbourhood before reaching the neighbourhood of another one, and

(iii) the probability of traversing the neighbourhoods of a given set of ω-limit sets in a particular order.

These quantifications are important in their own right as they help predict the performance of engineered systems, some of which are described in Section 1.1. We study the aforementioned metastability questions in Section 2.3. Such large time phenomena for diffusion processes with a small noise parameter have been studied in the past by Freidlin and Wentzell [37] under the "general position condition" (see [37, Sections 6.4-6.6]). Hwang and Sheu [44] studied the large time behaviour for diffusion processes under a more general setup. The key in both these works is the large deviation properties of the small noise diffusion processes over finite time durations, which have been established in [37, Chapter 5]. In this chapter, we extend the analysis to Markov mean-field jump processes, specifically $\{\mu^N(\cdot), N \geq 1\}$.

The proof of Theorem 2.1 is carried out using lower bounds (Theorem 2.8) for the probability that, starting from any point in $\mathcal{M}_1^N(\mathcal{Z})$, the process μ^N is in a small neighbourhood of one of the *most stable*[1] ω-limit set(s) of the McKean-Vlasov equation (2.1) when time is of the order $\exp\{N(\Lambda - \delta_0)\}$, for a small $\delta_0 > 0$. The constant Λ is defined using "costs of passages" between the ω-limit sets of the McKean-Vlasov equation (2.1). These costs are quantified in terms of the large deviations rate function associated with the process μ^N via certain graphs called W-graphs (see Section 2.3.2 for the definition of W-graphs). In particular, Λ is positive when the limiting dynamics (2.1) has multiple stable ω-limit sets. See (2.9) for a precise definition of Λ.

Our next result is, in a certain sense, a converse of Theorem 2.1. Let i_0 be one of the most stable ω-limit set(s) of (2.1).

Theorem 2.2. *There exist $\nu_0 \in \mathcal{M}_1(\mathcal{Z})$, $\delta > 0$, $\beta > 0$, $\rho_1 > 0$ and $N_0 \geq 1$ such that, with $T = \exp\{N(\Lambda - \delta)\}$,*

$$P_\nu(\mu^N(T) \in (\textit{the } \rho_1 \textit{ neighbourhood of } i_0)) \leq \exp\{-N\beta\}.$$

for all ν in the ρ_1-neighbourhood of ν_0 in $\mathcal{M}_1^N(\mathcal{Z})$ and $N \geq N_0$.

In other words, when time is of the order $\exp\{N(\Lambda - \delta)\}$, there are initial conditions $\nu \in \mathcal{M}_1^N(\mathcal{Z})$ such that the probability that $\mu^N(\exp\{N(\Lambda - \delta)\})$ is in a small neighbourhood of one of the most stable ω-limit set(s) is exponentially small. The process is then not likely to have equilibrated because it has not visited a set with high invariant measure. Thus, Theorem 2.1 and Theorem 2.2 together indicate that the constant Λ is sharp (in the exponential scale) for the time required for equilibration of $\mu^N(\cdot)$.

A convergence result similar to that of Theorem 2.1 for the mean-field discrete-time setting but without the specification of the constant Λ was established by Panageas and Vishnoi [68]. Let us reemphasise that our setting is a continuous-time setting. To identify the constant Λ in this setting, we must study the large deviation asymptotics in greater detail. Theorems 2.1 and 2.2 combine time and the number of particles. Additionally, Theorem 2.1 is a statement that holds uniformly over all initial conditions unlike the convergence bounds (over time) for a fixed number of particles with a given initial condition, e.g. [84]. The proof of Theorem 2.1 is inspired by that of Hwang and Sheu's [44, Theorem 2.1, Part I] where similar results are established for small noise diffusions.

[1] See Section 2.3.5 for a precise definition.

2.1.2.2 Asymptotics of the second largest eigenvalue

Our second main result is on the asymptotics of the second largest eigenvalue of the generator L^N of the Markov process μ^N when it is reversible with respect to its invariant measure $\wp^N$. That is, the operator L^N is self-adjoint in $L^2(\wp^N)$ and it admits a spectral expansion; let $0 = \lambda_1^N > -\lambda_2^N \geq -\lambda_3^N \geq \cdots$ denote its eigenvalues in the decreasing order. See Example 2.2 for the description of a reversible system that arises in statistical physics. For a fixed N, the convergence speed of the process μ^N to its invariant measure (over time) can be understood by studying the modulus of the second largest eigenvalue of L^N (i.e. λ_2^N). Using the results on the large time behaviour of μ^N and the convergence result in Theorem 2.1, we show that the modulus of the second largest eigenvalue of L^N scales as $\exp\{-N\Lambda\}$; here Λ (defined in (2.9)) is the constant that appears in the statement of Theorem 2.1. More precisely,

Theorem 2.3. *Assume that L^N is reversible with respect to $\wp^N$ for each $N \geq 1$. Then,*

$$\lim_{N\to\infty} \frac{1}{N} \log \lambda_2^N = -\Lambda.$$

It turns out that Λ can be positive only when there are metastable states in the limiting dynamics (2.1) (i.e. when (2.1) possesses multiple ω-limit sets). In such situations, one expects slower convergence to the invariant measure for large values of N. On the other hand, Λ can be 0, for example, when the limiting dynamics (2.1) has a unique globally asymptotically stable equilibrium; in this special case, convergence of μ^N to its invariant measure does not suffer from the slowing down phenomenon associated with positive Λ. In fact, Panageas and Vishnoi [68] and Panageas et al. [69] show that the mixing time is $O(\log N)$ in the discrete-time setting. Kifer [49] considers a more restrictive discrete-time model, which does not cover the mean-field model, and identifies the constant analogous to Λ [49, Theorem 4.3]. The restriction is that the state space of μ^N is the same for each N and that a certain uniform finite duration large deviation principle should hold with the rate function satisfying a continuity property. One can view our result as an extension of Kifer's [49, Theorem 4.3] to the continuous-time *mean-field* setting, where the state space of the Markov process μ^N changes with N. Hwang and Sheu [44] establish a result similar to ours on the scaling of the second largest eigenvalue of a reversible small noise diffusion process, and our method of proof is inspired by their approach.

2.1.2.3 Convergence to a global minimum via controlled addition of particles

Our third main result is on the convergence of the empirical measure process to a global minimum of a natural 'entropy' function when particles are injected over time at a specific rate reminiscent of the simulated annealing algorithm's cooling schedule, $N(t) = \lfloor \frac{\log(2+t)}{c^*+\delta} \rfloor$ for a suitable c^* and any $\delta > 0$. This entropy function is the large deviations rate function associated with the sequence of invariant measures $\{\wp^N, N \geq 1\}$, which is in turn defined in terms of the large deviations rate function associated with the process μ^N; see (2.8) for its definition.

Fix $c > 0$. Let $N_0 = \min\{n \in \mathbb{N} : \exp\{nc\} - 2 \geq 0\}$, $t_{N_0} = 0$, and for each $N > N_0$, let $t_N = \exp\{Nc\} - 2$. We construct a process with controlled addition of particles as follows. We start with N_0 particles with certain initial states and let the process evolve according to the generator L^{N_0} until time t_{N_0+1}. For each $N > N_0$, we add an extra particle at time t_N, and for a fixed state $z_0 \in \mathcal{Z}$, we set the state of the new particle to z_0 and let the process evolve according to the generator L^N from t_N to t_{N+1} (see a more precise description of the process in Section 2.5). Let $\bar{\mu}$ denote the above time-inhomogeneous Markov process and let $P_{0,\nu}$ denote the law of $\bar{\mu}$ on $D([0,\infty), \mathcal{M}_1(\mathcal{Z}))$ with initial condition $\bar{\mu}(0) = \nu$. Also, let L denote the set of all ω-limit sets of the McKean-Vlasov equation (see Section 2.3.1 for its precise definition) and let $\tilde{L}_0$ denote the set of all global minima of the entropy function (see Section 2.3.5.1 for the precise definition of $\tilde{L}_0$). Our convergence result is the following.

Theorem 2.4. *Assume that $\tilde{L}_0 \neq L$. There exists a constant $c^* > 0$ such that for all $c > c^*$ and any $\rho_1 > 0$,*

$$P_{0,\nu}(\bar{\mu}(t) \in (\textit{the } \rho_1\textit{-neighbourhood of } \tilde{L}_0)) \to 1$$

as $t \to \infty$, uniformly for all $\nu \in \mathcal{M}_1^{N_0}(\mathcal{Z})$.

Note that the convergence to a global minimum holds for all starting points. This is of use in situations where a population growth schedule is applied in order to *engineer* the mean-field system's movement to a desired equilibrium point, as time $t \to \infty$. One can also use this approach to study numerically the most likely region in which the process μ^N spends time for large values of N, under stationarity. Again, our proof is inspired by the analysis of the simulated annealing algorithm in [44, Part III]. We can also choose the transition rates of the particles so as to minimise a given "nice" function on $\mathcal{M}_1(\mathcal{Z})$, see Example 2.3.

2.1.3 Key ingredients for the proofs

The proofs of our main results follow the outlines in [44]. However, in order to make them work in our present context (which involves jump Markov processes and the mean-field setting), we need to establish the following properties:

- a uniform version of the finite-duration large deviation principle for $\{(\mu^N(t), t \in [0,T]), N \geq 1\}$, where the uniformity is over the initial condition;
- continuity of the cost function associated with movement between points on the simplex $\mathcal{M}_1(\mathcal{Z})$;
- strong Markov property of $\mu^N(\cdot)$.

The key insight from this chapter is the abstraction of these three properties and their importance in establishing the large time behaviour and metastability properties of mean-field systems. We leverage the results of [15] to establish the above properties.

We now describe the key ideas in the proofs of the main results of this chapter.

To prove Theorem 2.1, one possible approach is to wait long enough for the process μ^N to hit a neighbourhood of one of the most stable ω-limit set(s) of (2.1), regardless of the initial condition, and then allow sufficient additional time for the process to mix well. We prove Theorem 2.1 using this idea; we first consider a sequence of passages of μ^N between neighbourhoods of ω-limit sets of (2.1) to reach one of the most stable ω-limit set. Each of these passages take place between "stable" subsets of ω-limit sets called cycles (see Section 2.3.3). Probability of each of these passages over time intervals of the form $\exp\{N \times \text{constant}\}$ for appropriate constants can be lower bounded, thanks to the uniform large deviation property of μ^N (see Theorem 2.8). We then tie them up using the strong Markov property of μ^N. These steps yield a lower bound on the transition probability for μ^N (see Corollary 2.5) and Theorem 2.1 follows as a consequence of this. We can also produce an upper bound for probability of these passages for suitable initial conditions if enough time has not lapsed (see (2.11) in Theorem 2.8). Theorem 2.2 follows as a consequence of this upper bound.

Theorem 2.3 follows from an application of Theorem 2.1. We use the spectral expansion of the generator of μ^N, when it is reversible with respect to its invariant measure $\wp^N$, and the large deviation principle for $\{\wp^N, N \geq 1\}$ to prove Theorem 2.3.

In Theorem 2.4, to bring the process μ^N to one of the most stable ω-limit set(s) of (2.1) (i.e., one of the global minima of our entropy function), regardless of the initial condition, we introduce new particles over time in a controlled fashion. Before reaching a global minimum, the system may possibly explore other local minima. Since addition of the particles amounts to reduction of the "noise" in the process μ^N, we must make sure that the particles are introduced sufficiently slowly over time so that the system does not get trapped in a local minimum. This is achieved by the choice of our particle addition schedule $N(t), t \geq 0$, which is the analogue of the cooling schedule in simulated annealing. The schedule also enables us to apply the uniform large deviation principle over sufficiently long time durations to $\bar{\mu}$ so as to extend the results on the large time behaviour used in the proof of Theorem 2.1 to the present situation when the number of particles change over time (see Lemma 2.12–2.14). These extensions along with the method to analyse the passages of the system through cycles, the idea used in the proof of Theorem 2.1, enables us to prove a $1 - o(1)$ lower bound on the probability that $\bar{\mu}(t)$ belongs to a neighbourhood of a global minimum of our entropy function as $t \to \infty$, no matter where we start the process.

2.2 Preliminaries: Large deviations over finite time durations

In this section, we present a large deviation principle for the process μ^N over finite time durations. This result will be used later to study the large time behaviour of μ^N and the rate of convergence of μ^N to its invariant measure.

Fix $T > 0$. We introduce some notations. Let $p^{(N)}_{\nu^N,[0,T]}$ denote the solution to the $D([0,T], \mathcal{M}_1(\mathcal{Z}))$-valued martingale problem for L^N, i.e., the law of the empirical measure process $(\mu^N(t), t \in [0,T])$, and let $p^{(N)}_{\nu^N,T}$ denote the law of the terminal-time empirical measure $\mu^N(T) \in \mathcal{M}_1(\mathcal{Z})$, with a deterministic initial condition $\mu^N(0) = \nu^N$. Let $\mathcal{AC}[0,T]$ denote the space of absolutely continuous $\mathcal{M}_1(\mathcal{Z})$-valued paths on $[0,T]$ (in particular they are differentiable for almost all $t \in [0,T]$; see [55, Definition 3.1]). Define

$$\tau^*(u) \coloneqq \begin{cases} \infty & \text{if } u < -1 \\ 1 & \text{if } u = -1 \\ (u+1)\log(u+1) - u & \text{if } u > -1, \end{cases}$$

which is the Fenchel-Legendre transform of $\tau(u) = e^u - u - 1, u \in \mathbb{R}$. Recall the definition of the family of rate matrices $(\Lambda_\xi, \xi \in \mathcal{M}_1(\mathcal{Z}))$ from Section 2.1.2. We have the following large deviation principle (LDP) for the sequence $\{p^{(N)}_{\nu^N,[0,T]}, N \geq 1\}$ on $D([0,T], \mathcal{M}_1(\mathcal{Z}))$ (see [55, Theorem 3.1], [15, Theorem 3.2]).

Theorem 2.5. *Suppose that the initial conditions $\nu^N \to \nu$ in $\mathcal{M}_1(\mathcal{Z})$ as $N \to \infty$. Then the sequence of probability measures $\{p^{(N)}_{\nu^N,[0,T]}, N \geq 1\}$ on the space $D([0,T], \mathcal{M}_1(\mathcal{Z}))$ satisfies the LDP with rate function $S_{[0,T]}(\cdot|\nu)$ defined as follows. If $\mu(0) = \nu$ and $\mu \in \mathcal{AC}[0,T]$, then*

$$S_{[0,T]}(\mu|\nu) = \int_{[0,T]} \sup_{\alpha \in \mathbb{R}^{\mathcal{Z}}} \left\{ \sum_{z \in \mathcal{Z}} \alpha(z)(\dot{\mu}_t(z) - \Lambda^*_{\mu_t}\mu_t(z)) \right.$$

$$- \sum_{(z,z')\in\mathcal{E}} \tau(\alpha(z') - \alpha(z))\lambda_{z,z'}(\mu_t)\mu_t(z) \Bigg\} dt,$$

and $S_{[0,T]}(\mu|\nu) = +\infty$ otherwise. Moreover, if $S_{[0,T]}(\mu|\nu) < \infty$, then there exists a unique family of rate matrices $L(t) = (l_{z,z'}(t), z, z' \in \mathcal{Z}), t \in [0,T]$, such that $t \mapsto L(t)$ is measurable, μ is the solution to

$$\dot{\mu}(t) = L(t)^*\mu(t),\ t \in [0,T],\ \mu(0) = \nu,$$

and

$$S_{[0,T]}(\mu|\nu) = \int_{[0,T]} \sum_{(z,z')\in\mathcal{E}} \mu(t)(z)\lambda_{z,z'}(\mu(t))\tau^* \left(\frac{l_{z,z'}(t)}{\lambda_{z,z'}(\mu(t))} - 1 \right) dt,$$

where $L(t)^*$ denotes the transpose of $L(t)$, $t \in [0,T]$.

We can interpret the rate function $S_{[0,T]}$ as follows. Starting at ν^N, the process μ^N is likely to be in the neighbourhood of the solution to the McKean-Vlasov equation (2.1) with initial condition ν (with high probability). In order for the process μ^N to be in the neighbourhood of some other path, we need to apply a control given by the rate matrix L; $S_{[0,T]}(\mu|\nu)$ is the cost of this control. In particular, since the solution to the McKean-Vlasov equation starting at ν has zero-cost (i.e. $S_{[0,T]}(\mu_\nu|\nu) = 0$ where μ_ν denotes the solution to (2.1) starting at ν), the limiting behaviour that $\mu^N(\cdot) \xrightarrow{P} \mu_\nu(\cdot)$ in $D([0,T], \mathcal{M}_1(\mathcal{Z}))$ as $N \to \infty$ follows. See [30] for some remarks about the form of the rate function and for another representation of the rate function in terms of a relative entropy.

Here is an outline of the proof of Theorem 2.5: one looks at a system of non-interacting particles where the transition rates of a particle do not depend on the empirical measure, and considers the corresponding empirical measure process over $[0,T]$. Since at most one particle can jump at a given point of time, the measure $p^{(N)}_{\nu^N,[0,T]}$ is absolutely continuous with respect to the measure corresponding to the above non-interacting system on $D([0,T], \mathcal{M}_1(\mathcal{Z}))$. One can then write the Radon-Nikodym derivative using the Girsanov formula and show continuity properties of the same. An application of an extension of Sanov's theorem (see [26, Theorem 3.5]) tells us that the non-interacting particle system obeys the LDP on $D([0,T], \mathcal{M}_1(\mathcal{Z}))$. The above theorem then follows by an application of Varadhan's integral lemma (see [29, Theorem 4.3.1]). This approach has been carried out for a system of interacting diffusions in [26] and for jump processes in [55, 15]. One can also prove various special cases of Theorem 2.5 via other simpler methods; for example, for fixed initial conditions, i.e., when $\nu^N = \delta_z$ for some $z \in \mathcal{Z}$ and for all $N \geq 1$, one can use a modification of Varadhan's lemma to obtain the LDP for $p^{(N)}_{\delta_z,[0,T]}$ (see [27]), but letting the initial condition to be arbitrary, except for the constraint $\nu^N \to \nu$ weakly, is crucial to obtain a uniform version of the Theorem 2.5 (see Corollary 2.1), which is used to prove our main results.

We now recall a theorem that gives the large deviation principle for the sequence $\{p^{(N)}_{\nu^N,T}, N \geq 1\}$ on $\mathcal{M}_1(\mathcal{Z})$. This can be obtained from the above theorem by an application of the contraction principle to the coordinate projection map $D([0,T],\mathcal{M}_1(\mathcal{Z})) \ni \mu \mapsto \mu(T)$ (see [29, Theorem 4.2.1], [15, Theorem 3.3]).

Theorem 2.6. *Suppose that the initial conditions $\nu^N \to \nu$ in $\mathcal{M}_1(\mathcal{Z})$ as $N \to \infty$. Then the sequence of probability measures $\{p^{(N)}_{\nu^N,T}, N \geq 1\}$ on the space $\mathcal{M}_1(\mathcal{Z})$ satisfies the LDP with the rate function*

$$S_T(\xi|\nu) := \inf\{S_{[0,T]}(\mu|\nu) : \mu(0) = \nu, \mu(T) = \xi, \mu \in \mathcal{AC}[0,T]\}.$$

Moreover, the above infimum is attained, i.e., there exists a path $\hat{\mu} \in \mathcal{AC}[0,T]$ such that $\hat{\mu}(0) = \nu$, $\hat{\mu}(T) = \xi$ and $S_{[0,T]}(\hat{\mu}|\nu) = S_T(\xi|\nu)$.

Here, $S_T(\xi|\nu)$ can be interpreted as the minimum cost of passage from the profile ν to the profile ξ in time T, among all paths from ν to ξ in time T. It can be shown that S_T is continuous on $\mathcal{M}_1(\mathcal{Z}) \times \mathcal{M}_1(\mathcal{Z})$ by constructing piecewise constant velocity trajectories between points on $\mathcal{M}_1(\mathcal{Z})$ (see [15, Lemma 3.3]).

We also have the following *uniform* LDP for the sequence $\{p^{(N)}_{\nu^N,[0,T]}, N \geq 1\}$ (see [15, Corollary 3.1]) when the initial condition is allowed to lie in a compact set.

Corollary 2.1. *For any compact set $K \subset \mathcal{M}_1(\mathcal{Z})$, any closed set $F \subset D([0,T],\mathcal{M}_1(\mathcal{Z}))$, and any open set $G \subset D([0,T],\mathcal{M}_1(\mathcal{Z}))$, we have*

$$\limsup_{N\to\infty} \frac{1}{N} \log \sup_{\nu^N \in K\cap\mathcal{M}_1^N(\mathcal{Z})} p^{(N)}_{\nu^N,[0,T]}(\mu^N \in F) \leq -\inf_{\nu\in K}\inf_{\mu\in F} S_{[0,T]}(\mu|\nu), \tag{2.2}$$

and

$$\liminf_{N\to\infty} \frac{1}{N} \log \inf_{\nu^N \in K\cap\mathcal{M}_1^N(\mathcal{Z})} p^{(N)}_{\nu^N,[0,T]}(\mu^N \in G) \geq -\sup_{\nu\in K}\inf_{\mu\in G} S_{[0,T]}(\mu|\nu). \tag{2.3}$$

For a proof of the above, see [29, Corollary 5.6.15]. Note that, since the space $\mathcal{M}_1(\mathcal{Z})$ is compact, we may take $K = \mathcal{M}_1(\mathcal{Z})$ in the above corollary.

Remark 2.1. The version of uniform LDP presented in Corollary 2.1 is slightly different from the definition of uniform LDP in Freidlin and Wentzell [37, Section 3, Chapter 3]. The version presented here suffices for the proofs of our main results since our state space $\mathcal{M}_1(\mathcal{Z})$ is compact and the rate function S_T defined in Theorem 2.6 is continuous (see [77, Theorem 2.7] and [15, Appendix A]).

2.3 Large time behaviour

In the study of the large time behaviour of μ^N, an important role is played by the Freidlin-Wentzell quasipotential $V : \mathcal{M}_1(\mathcal{Z}) \times \mathcal{M}_1(\mathcal{Z}) \to [0, \infty)$ defined by

$$V(\nu, \xi) \coloneqq \inf\{S_{[0,T]}(\mu|\nu) : \mu(T) = \xi, T > 0\},$$

i.e., $V(\nu, \xi)$ denotes the minimum cost of transport from ν to ξ in an arbitrary but finite time.

We say that $\nu \sim \xi$ (ν is equivalent to ξ) if $V(\nu, \xi) = 0$ and $V(\xi, \nu) = 0$. It is easy to see that $\sim$ defines an equivalence relation on $\mathcal{M}_1(\mathcal{Z})$. To study the large time behaviour of the process μ^N, we make the following assumptions on the McKean-Vlasov equation (2.1) (see [37, Chapter 6, Section 2, Condition A]):

(B1) There exists a finite number of compact sets $K_1, K_2, \dots, K_l$ such that

- For each $i = 1, 2, \dots l$, $\nu_1, \nu_2 \in K_i$ implies $\nu_1 \sim \nu_2$.
- For each $i \neq j$, $\nu_1 \in K_i$ and $\nu_2 \in K_j$ implies $\nu_1 \nsim \nu_2$.
- Every ω-limit set of the dynamical system (2.1) lies completely in one of the compact sets K_i.

Since $V(\nu_1, \nu_2) = 0$ whenever $\nu_1, \nu_2 \in K_i$ for any $1 \leq i \leq l$, we can define

$$V(K_i, K_j) \coloneqq \inf\{S_{[0,T]}(\mu|\nu) : \nu \in K_i, \mu(T) \in K_j, T > 0\},$$

which is interpreted as the minimum cost of going from K_i to K_j. We also define the minimum cost of going from K_i to K_j without touching the other compact sets $K_k, k \neq i, j$ by

$$\tilde{V}(K_i, K_j) \coloneqq \inf\{S_{[0,T]}(\mu|\nu) : \nu \in K_i, \mu(t) \notin \cup_{k \neq i,j} K_k \text{ for all } t \in [0, T], \mu(T) \in K_j, T > 0\}. \tag{2.4}$$

Using the definition of the rate function S_T, note that

$$V(\nu,\xi) = \inf_{T>0} S_T(\xi|\nu) \quad \text{and} \quad V(K_i,K_j) = \inf_{\nu\in K_i,\xi\in K_j} V(\nu,\xi).$$

Example 2.1. We provide two example where (B1) is satisfied.

1. (Wireless local area network). Let $\mathcal{Z} = \{0,1\}$. The edgeset $\mathcal{E}$ consists of the edges $(0,1)$ and $(1,0)$. Define the transition rates

$$\lambda_{z,z'}(\xi) = \begin{cases} c_0(1-\exp\{-(c_0\xi(0)+c_1\xi(1))\}) & \text{if } z=0, z'=1, \\ c_1 & \text{if } z=1, z'=0, \end{cases}$$

   where $c_0, c_1 > 0$. The limiting dynamics (2.1) is a one-dimensional ODE and it is given by

$$\dot{\mu}_t(0) = -c_0\mu_t(0)(1-\exp\{-(c_0\mu_t(0)+c_1(1-\mu_t(0)))\}) + c_1(1-\mu_t(0)),\ t\geq 0.$$

   Let $f(x) = -c_0x(1-\exp\{-(c_0x+c_1(1-x))\}) + c_1(1-x)$, $x\in[0,1]$. Note that $f(0)>0$ and $f(1)<0$. It is easy to check that, if $c_0>c_1$, then $f'(x)<0$ for all $x\in(0,1)$. As a consequence, there exists a unique $\xi^*\in\mathcal{M}_1(\mathcal{Z})$ such that all trajectories of the above dynamical system converge to $\xi^*(0)$. Thus, assumption (B1) holds with $l=1$, $K_1=\{\xi^*\}$.

2. (Dynamic alternate routing in loss networks). Fix $C\in\mathbb{Z}_+$ and let $\mathcal{Z}=\{0,1,\ldots,C\}$. The edgeset $\mathcal{E}$ consists of the forward edges $\{(z,z+1), 0\leq z\leq C-1\}$ and the backward edges $\{(z,z-1), 1\leq z\leq C\}$. For $(z,z')\in\mathcal{E}$ and $\xi\in\mathcal{M}_1(\mathcal{Z})$, define the transition rates

$$\lambda_{z,z'}(\xi) = \begin{cases} z & \text{if } z\neq 0, z'=z-1, \\ \alpha+\alpha\xi(C)\times 2(1-\xi(C)) & \text{if } z\neq C, z'=z+1, \end{cases}$$

   where $\alpha>0$. This model arises in the context of dynamic alternate routing in loss networks. For certain values of α, the limiting ODE (2.1) possesses two stable equilibria (say ξ_1^* and ξ_2^*) and an unstable equilibrium (say ξ_3^*) [40, 66]. Thus, assumption (B1) is satisfied with $l=3$, $K_i=\{\xi_i^*\}$, $i=1,2,3$.

For a model of malware propagation where a limit cycle and an unstable equilibrium arise, see Benaïm and Le Boudec [6, Section 4.1].

2.3.1 Preliminary results

It turns out that, under assumption (B1), the large time behaviour of the process μ^N can be studied via a discrete time Markov chain whose state space is the union of small neighbourhoods of the compact sets $K_i, 1 \leq i \leq l$. To study this chain, we introduce some notation. Let $L = \{1, 2, \dots, l\}$. Given $0 < \rho_1 < \rho_0$, let γ_i (resp. Γ_i) denote the ρ_1-open neighbourhood (resp. ρ_0-open neighbourhood) of K_i. Let $\gamma = \cup_{i=1}^l \gamma_i$, $\Gamma = \cup_{i=1}^l \Gamma_i$, and $C = \mathcal{M}_1(\mathcal{Z}) \setminus \overline{\Gamma}$. For a set $A \subset \mathcal{M}_1(\mathcal{Z})$ and $\delta > 0$, let $[A]_\delta$ denote the δ-open neighbourhood of A, and for a subset $W \subset L$, abusing notation, let $[W]_\delta$ denote the δ-open neighbourhood of $\cup_{i \in W} K_i$. For each $n \geq 1$, we define the sequence of stopping times: $\tau_0 := 0$, $\sigma_n := \inf\{t > \tau_{n-1} : \mu^N(t) \in C\}$, $\tau_n := \inf\{t > \sigma_n : \mu^N(t) \in \gamma\}$, and define $Z_n^N := \mu^N(\tau_n)$. Since μ^N is strong Markov, Z^N is a discrete time Markov chain, and $Z_n^N \in \gamma \cap \mathcal{M}_1^N(\mathcal{Z})$ for all $n \geq 1$. For a measurable set $A \in \mathcal{M}_1(\mathcal{Z})$, we define the stopping time $\tau_A := \inf\{t > 0 : \mu^N(t) \notin A\}$, which denotes the time of first exit from the set A. Finally, for a subset $W \subset L$, we define the stopping time $\hat{\tau}_W := \inf\{t > 0 : \mu^N(t) \in \cup_{i \in W} \gamma_i\}$, and $\bar{\tau}_W := \inf\{t > 0 : \mu^N(t) \in \cup_{i \in L \setminus W} \gamma_i\}$, which denote the time of entry into the ρ_1-neighbourhood of W and the time of entry into the ρ_1-neighbourhood of $L \setminus W$, respectively.

We now state some results on the behaviour of the exit time from certain sets, which will be used in this chapter subsequently. These results are known in the case of both Markov jump processes as well as diffusion processes; see [15, Appendix], and [37, Chapter 6, Section 2]. The main ingredients that are used in proving these results are (i) the strong Markov property of the μ^N process, (ii) Theorem 2.5 and Corollary 2.1 on the LDP for finite time durations, and (iii) the joint continuity of the terminal time rate function $S_T(\cdot|\cdot)$ (see [15, Lemma 3.3]). Recall that P_ν denotes the law of $(\mu^N(t), t \geq 0)$ with initial condition $\mu^N(0) = \nu$ and E_ν denotes the corresponding expectation.

Lemma 2.1 ([15, Lemma A.3]). *Let $K \subset \mathcal{M}_1(\mathcal{Z})$ be a compact set such that all points in K are equivalent to each other (i.e., $\nu_1 \sim \nu_2$ for all $\nu_1, \nu_2 \in K$), and $K \neq \mathcal{M}_1(\mathcal{Z})$. Then, given $\varepsilon > 0$, there exist $\delta > 0$ and $N_0 \geq 1$ such that for all $N \geq N_0$ and $\nu \in [K]_\delta \cap \mathcal{M}_1^N(\mathcal{Z})$,*

$$E_\nu \tau_{[K]_\delta} \leq \exp\{N\varepsilon\}.$$

Lemma 2.2 ([15, Lemma A.4]). *Let $K \subset \mathcal{M}_1(\mathcal{Z})$ be a compact set and G be a neighbourhood of K. Then, given $\varepsilon > 0$, there exist $\delta > 0$ and $N_0 \geq 1$ such that for all $\nu \in \overline{[K]_\delta} \cap \mathcal{M}_1^N(\mathcal{Z})$*

and $N \geq N_0$

$$E_\nu\left(\int_0^{\tau_G} \mathbf{1}_{\{\mu^N(t)\in\overline{[K]_\delta}\}} dt\right) \geq \exp\{-N\varepsilon\}.$$

Lemma 2.3 ([15, Lemma A.5])**.** *Let $K \subset \mathcal{M}_1(\mathcal{Z})$ be a compact set that does not contain any ω-limit set of (2.1) entirely. Then, there exist positive constants c, T_0, and $N_0 \geq 1$ such that for all $T \geq T_0$, $N \geq N_0$, and $\nu \in K \cap \mathcal{M}_1^N(\mathcal{Z})$, we have*

$$P_\nu(\tau_K \geq T) \leq \exp\{-Nc(T - T_0)\}.$$

Corollary 2.2. *Under the conditions of Lemma 2.3, there exist $C > 0$ and $N_0 \geq 1$ such that for all $\nu \in K \cap \mathcal{M}_1^N(\mathcal{Z})$ and $N \geq N_0$,*

$$E_\nu \tau_K \leq C.$$

Recall the definition of the discrete time Markov chain Z^N on $\gamma \cap \mathcal{M}_1^N(\mathcal{Z})$. The next lemma gives upper and lower bounds on the one-step transition probabilities of the chain Z^N. These estimates play an important role in the study of the large time behaviour of the process μ^N, as we shall see in the sequel.

Lemma 2.4 ([15, Lemma A.6])**.** *Given $\varepsilon > 0$, there exist $\rho_0 > 0$ and $N_0 \geq 1$ such that, for any $\rho_2 < \rho_0$, there exists $\rho_1 < \rho_2$ such that for any $\nu \in [K_i]_{\rho_2} \cap \mathcal{M}_1^N(\mathcal{Z})$ and $N \geq N_0$, the one-step transition probability of the chain Z^N satisfies*

$$\exp\{-N(\tilde{V}(K_i, K_j) + \varepsilon)\} \leq P(\nu, \gamma_j) \leq \exp\{-N(\tilde{V}(K_i, K_j) - \varepsilon)\}. \tag{2.5}$$

Remark 2.2. In the above statement, $P(\nu, \gamma_j)$ is defined as $P(\nu, \gamma_j) := P_\nu(Z_1^N \in \gamma_j) = P_\nu(\mu^N(\tau_1) \in \gamma_j)$.

The key ingredient in the proof of the above lemma is Corollary 2.1 on the uniform large deviation principle on bounded sets. For the lower bound, one constructs a specific trajectory from ν to K_j and examines its cost. For the upper bound, one uses the strong Markov property at the hitting time of $[L]_{\rho_1}$ and the uniform large deviation principle. For details, the reader is referred to the proof of [15, Lemma A.6] for the case of Markov jump processes, and the proof of [37, Lemma 2.1, page 152] for the case of small noise diffusions.

2.3.2 Behaviour near attractors indexed by subsets of L

We now recall some results on the behaviour of the process μ^N near a small neighbourhood of attractors indexed by a given subset of L. Let $W \subset L$, $W \neq \emptyset$. A W-graph is a directed graph on L such that (i) each element of $L \setminus W$ has exactly one outgoing arrow and (ii) there are no closed cycles in the graph. We denote the set of W-graphs by $G(W)$. For each $i \in L$, we denote $G(\{i\})$ by $G(i)$. For a W-graph g, define

$$\tilde{V}(g) = \sum_{(m \to n) \in g} \tilde{V}(K_m, K_n). \tag{2.6}$$

If g does not have any edge (e.g., when L is a singleton), we use the convention $\tilde{V}(g) = 0$. Note that, using the estimate (2.5), $\tilde{V}$ can be used to estimate the probability that the process μ^N traverses through a sequence of neighbourhoods in the order specified by the graph g.

For $i \in L \setminus W$ and $j \in W$, let $G_{i,j}(W)$ denote the set of W-graphs in which there is a sequence of arrows leading from i to j. Define

$$I_{i,j}(W) \coloneqq \min\{\tilde{V}(g) : g \in G_{i,j}(W)\} - \min\{\tilde{V}(g) : g \in G(W)\}.$$

We recall the following result on the probability that the first entry of μ^N into a neighbourhood of a set $W \subset L$ takes place via a given compact set K_j, starting from a neighbourhood of K_i.

Lemma 2.5. *Let $W \subset L$, and let $i \in L \setminus W$ and $j \in W$. Given $\varepsilon > 0$, there exist $\rho > 0$ and $N_0 \geq 1$ such that for any $\rho_1 \leq \rho$, $\nu \in \gamma_i \cap \mathcal{M}_1^N(\mathcal{Z})$, and $N \geq N_0$, we have*

$$\exp\{-N(I_{i,j}(W) + \varepsilon)\} \leq P_\nu(\mu^N(\hat{\tau}_W) \in \gamma_j) \leq \exp\{-N(I_{i,j}(W) - \varepsilon)\}.$$

Proof. The proof of [37, Lemma 3.3, page 159] holds verbatim, by making use of the estimates in Lemma 2.4. □

Remark 2.3. While the above lemma provides an estimate of the probability $P_\nu(\mu^N(\hat{\tau}_W) \in \gamma_j)$, it does not provide any information about the sequence of states in L visited by the process μ^N while traversing from i to j. The latter can be understood via studying the minimisations in the definition of $I_{i,j}$, see [38].

Our next step is to understand the mean entry time $E_\nu \hat{\tau}_W$. For this, we need the following estimate on the stopping time τ_1; see [44, Lemma 1.3, Part I] for a similar estimate for small noise diffusion processes.

Lemma 2.6. *Given $\varepsilon > 0$, there exist $\rho_1 > 0$ and $N_0 \geq 1$ such that, for any $\nu \in \gamma \cap \mathcal{M}_1^N(\mathcal{Z})$ and $N \geq N_0$, we have*

$$E_\nu \tau_1 \leq \exp\{N\varepsilon\}.$$

Proof. With a sufficiently small $\rho_1 > 0$ to be chosen later, let $\rho_0 = 2\rho_1$ so that $[K_i]_{\rho_0}$ does not intersect with $[K_j]_{\rho_0}$ for all $j \neq i$. Note that, for any $\nu \in \gamma$,

$$E_\nu \tau_1 = E_\nu \sigma_1 + E_\nu(\tau_1 - \sigma_1).$$

Consider the first term. By Lemma 2.1, there exist $\rho > 0$ and $N_0 \geq 1$ such that for all $\rho_1 \leq \rho$, $\nu \in \gamma \cap \mathcal{M}_1^N(\mathcal{Z})$ and $N \geq N_0$, we have

$$E_\nu \sigma_1 \leq \exp\{N\varepsilon/2\}.$$

Let $F = \mathcal{M}_1(\mathcal{Z}) \setminus \gamma$. By the strong Markov property, the second term is

$$E_\nu(\tau_1 - \sigma_1) = E_{\mu^N(\sigma_1)}(\tau_F).$$

Therefore, it suffices to estimate $E_{\nu'} \tau_F$ for $\nu' \in F$. Since the compact set F does not contain any ω-limit set, by Corollary 2.2, there exist a constant $C > 0$ and $N_1 \geq N_0$ such that for any $\nu' \in F \cap \mathcal{M}_1^N(\mathcal{Z})$

$$E_{\nu'} \tau_F \leq C.$$

This completes the proof of the lemma. □

Define

$$I_i(W) \coloneqq \min\{\tilde{V}(g) : g \in G(W)\} \\ - \min\{\tilde{V}(g) : g \in G(W \cup \{i\}) \text{ or } g \in G_{i,j}(W \cup \{j\}), i \neq j, j \in L \setminus W\}.$$

The next lemma is about the mean entry time into a neighbourhood of a given set $W \subset L$ starting from a neighbourhood of K_i; see [44, Lemma 1.6, Part I] for a similar estimate on small noise diffusion processes.

Lemma 2.7. *Let $W \subset L$, and let $i \in L \setminus W$. Given $\varepsilon > 0$, there exist $\rho > 0$ and $N_0 \geq 1$ such that for any $\rho_1 \leq \rho$, $\nu \in \gamma_i \cap \mathcal{M}_1^N(\mathcal{Z})$, and $N \geq N_0$, we have*

$$\exp\{N(I_i(W) - \varepsilon)\} \leq E_\nu \hat{\tau}_W \leq \exp\{N(I_i(W) + \varepsilon)\}.$$

Proof. We first prove the upper bound. Note that, by the strong Markov property, we have

$$E_\nu \hat{\tau}_W = E_\nu \tau_v \leq \sum_{m=1}^{\infty} E_\nu \left(\mathbf{1}_{\{v=m\}} \times m \sup_{\nu' \in \gamma} E_{\nu'} \tau_1 \right),$$

where v is the hitting time of the chain Z_n^N on the set W. Using Lemma 2.6 and the upper bound on $E_\nu v$ derived in [37, Lemma 3.4, page 162], for sufficiently small ρ_1 and sufficiently large N, we have that

$$E_\nu \hat{\tau}_W \leq \exp\{N(I_i(W) + \varepsilon)\}$$

holds for all $\nu \in \gamma_i \cap \mathcal{M}_1^N(\mathcal{Z})$. For the lower bound, Lemma 2.2 implies that, for all sufficiently small ρ_1 and sufficiently large N, we have that

$$E_\nu \tau_1 \geq \exp\{-N\varepsilon\}$$

holds for all $\nu \in \gamma$. Also,

$$E_\nu \hat{\tau}_W = E_\nu \tau_v \geq \sum_{m=1}^{\infty} E_\nu \left(\mathbf{1}_{\{v=m\}} \times m \inf_{\nu' \in \gamma} E_{\nu'} \tau_1 \right),$$

hence, using the lower bound on $E_\nu v$ derived in [37, Lemma 3.4, page 162], we get

$$E_\nu \hat{\tau}_W \geq \exp\{N(I_i(W) - \varepsilon)\}$$

for all $\nu \in \gamma_i \cap \mathcal{M}_1^N(\mathcal{Z})$ and sufficiency large N. □

2.3.3 Cycles

We now define the notion of cycles, which helps us to describe the most probable way in which the process μ^N, for large N, traverses neighbourhoods of various compact sets K_i, and the time required to go from one to another. Recall the definition of $\tilde{V}$ from (2.4). We interpret $\tilde{V}(K_i, K_j)$ as the "communication cost" from i to j. Define $\tilde{V}(K_i) \coloneqq \min_{j \neq i} \tilde{V}(K_i, K_j)$. We say that $i \to j$ if $\tilde{V}(K_i) = \tilde{V}(K_i, K_j)$. For $j \neq i$, the probability that μ^N hits a small neighbourhood of K_j upon exit from a small neighbourhood of K_i is of the form $\exp\{-N(\tilde{V}(K_i, K_j) - \tilde{V}(K_i))\}$, and the mean exit time from a small neighbourhood of K_i is of the form $\exp\{N\tilde{V}(K_i)\}$ [37, Chapter 6, Section 5]. In particular, the indices that attain the minimum above are the most likely sets that will be visited by the process μ^N, for large enough N, starting from a neighbourhood of K_i. For $i, j \in L$, we say that $i \Rightarrow j$ if there exists a sequence of arrows leading from i to j, i.e., there exists $i_1, i_2, \ldots, i_n$ in L such that $i \to i_1 \to i_2 \to \cdots \to i_n \to j$. Again, the above sequence of arrows from i to j is one among the *locally* most likely sequences in which the process traverses from a neighbourhood of K_i to that of K_j for large N.

Definition 2.1. A 1-cycle π is a directed graph on a subset of elements of L satisfying

1. $i \in \pi$ and $i \Rightarrow j$ implies $j \in \pi$.
2. For any $i \neq j$ in π, we have $i \Rightarrow j$ and $j \Rightarrow i$.

That is, a 1-cycle is a subset of the elements of L along with a certain assignment of arrows among them according to the numbers $\tilde{V}(\cdot, \cdot)$. For example, if $L = \{1, 2, 3\}$, $1 \to 2$, $2 \to 1$, and $3 \to 1$ are the only possible arrows (i.e., $\tilde{V}(K_1) = \tilde{V}(K_1, K_2) < \tilde{V}(K_1, K_3)$, $\tilde{V}(K_2) = \tilde{V}(K_2, K_1) < \tilde{V}(K_2, K_3)$, and $\tilde{V}(K_3) = \tilde{V}(K_3, K_1) < \tilde{V}(K_3, K_2)$), then the graph on $\{1, 2\}$ consisting of the arrows $1 \to 2$ and $2 \to 1$ is a 1-cycle. The set $\{3\}$ is not part of a 1-cycle. It can be shown that there exists a 1-cycle (see the proof of [44, Lemma 1.9, Part I]).

We now define cycle of 1-cycles. Let $L_0 = L$. Define

$$L_1 \coloneqq \{\pi : \pi \text{ is a cycle in } L\} \cup \{i \in L : i \text{ is not in any cycle}\}.$$

That is, the elements of L_1 are either 1-cycles in L or elements of L that do not belong to any 1-cycle. In the previous example, L_1 is the set $\{\{1, 2\}\} \cup \{\{3\}\}$. Ultimately, we view the elements of L_1 as subsets of L. If $\pi \in L_1$, we write $K \in \pi$ to indicate that the index of the compact set K in $\{1, 2, \ldots, l\}$ is an element of π. We now proceed to define the "communicating cost" between the elements of L_1. For $\pi_1, \pi_2 \in L_1$, $\pi_1 \neq \pi_2$, define

$$\hat{V}(\pi_1) := \max\{\tilde{V}(K) : K \in \pi_1\},$$

$$\tilde{V}(\pi_1, \pi_2) := \hat{V}(\pi_1) + \min\{\tilde{V}(K_1, K_2) - \tilde{V}(K_1) : K_1 \in \pi_1, K_2 \in \pi_2\},$$

and

$$\tilde{V}(\pi_1) := \min\{\tilde{V}(\pi_1, \pi_2) : \pi_2 \in L_1, \pi_2 \neq \pi_1\}.$$

That is, $\tilde{V}(\pi_1, \pi_2)$ is the communication cost from π_1 to π_2, and it generalises the quantity $\tilde{V}(K_i, K_j)$ to 1-cycles. Similarly, $\tilde{V}(\pi_1)$ generalises the quantity $\tilde{V}(K_i)$ to 1-cycles. If π_1, π_2 are 1-cycles, $\pi_1 \neq \pi_2$, then upon exit from a small neighbourhood of the elements of π_1, the probability that the process μ^N enters a small neighbourhoods of the elements of π_2 is of the form $\exp\{-N(\tilde{V}(\pi_1, \pi_2) - \tilde{V}(\pi_1))\}$, and the mean exit time from a small neighbourhood of the elements of π_1 is of the form $\exp\{N\tilde{V}(\pi_1)\}$. We say that $\pi_1 \to \pi_2$ if $\tilde{V}(\pi_1) = \tilde{V}(\pi_1, \pi_2)$, and we say that $\pi_1 \Rightarrow \pi_2$ if there is a sequence of arrows leading from π_1 to π_2. This gives a cycle of 1-cycles, which we call 2-cycles.

Let us now define the hierarchy of cycles. Having defined $(m-1)$-cycles and the sets $L_0, L_1, \ldots, L_{m-2}$, we define m-cycles as follows. Note that

$$L_{m-1} = \{\pi^{m-1} : \pi^{m-1} \text{ is an } (m-1)\text{-cycle}\} \cup \{\pi^{m-2} \in L_{m-2} : \pi^{m-2} \text{ is not in any } (m-1)\text{-cycle}\}.$$

That is, the elements of L_{m-1} are either $(m-1)$-cycles or elements of L_{m-2} that are not part of any $(m-1)$-cycle; in both cases, they are ultimately viewed as subsets of L. Given $\pi^{m-1} \in L_{m-1}$ and an $(m-2)$-cycle π^{m-2}, we write $\pi^{m-2} \in \pi^{m-1}$ if the elements of π^{m-2} (when it is viewed as a subset of L) are part of π^{m-1}. For $\pi^{m-1} \in L_{m-1}$, define

$$\hat{V}(\pi^{m-1}) := \max\{\tilde{V}(\pi^{m-2}) : \pi^{m-2} \in \pi^{m-1}\},$$

$$\tilde{V}(\pi_1^{m-1}, \pi_2^{m-1}) := \hat{V}(\pi_1^{m-1}) + \min\{\tilde{V}(\pi_1^{m-2}, \pi_2^{m-2}) - \tilde{V}(\pi_1^{m-2}) : \pi_1^{m-2} \in \pi_1^{m-1}, \pi_2^{m-2} \in \pi_2^{m-1}\},$$

and

$$\tilde{V}(\pi_1^{m-1}) := \min\{\tilde{V}(\pi_1^{m-1}, \pi_2^{m-1}) : \pi_2^{m-1} \in L_{m-1}, \pi_2^{m-1} \neq \pi_1^{m-1}\}.$$

We say that $\pi_1^{m-1} \to \pi_2^{m-1}$ if $\tilde{V}(\pi_1^{m-1}) = \tilde{V}(\pi_1^{m-1}, \pi_2^{m-1})$. We have

Definition 2.2. An m-cycle π^m is a directed graph on a subset of elements of L_{m-1} satisfying

1. For $\pi_1^{m-1}, \pi_2^{m-1} \in L_{m-1}$, $\pi_1^{m-1} \in \pi^m$ and $\pi_1^{m-1} \Rightarrow \pi_2^{m-1}$ implies $\pi_2^{m-1} \in \pi^m$.
2. For any $\pi_1^{m-1}, \pi_2^{m-1} \in \pi^m$, we have $\pi_1^{m-1} \Rightarrow \pi_2^{m-1}$ and $\pi_2^{m-1} \Rightarrow \pi_1^{m-1}$.

If we continue this way, for some $m \geq 1$, the set L_m will eventually be a singleton, at which point we stop. See [86] for a numerical example that consists of three 1-cycles and a 2-cycle when L has 9 elements.

We now state some results on the mean exit time from a cycle and the most probable cycle the process μ^N visits upon exit from a given cycle. For convenience, the set of elements of L constituting a k-cycle π^k (through the hierarchy of cycles) is also denoted by π^k. Also, for $W \subset L$, we define $\gamma_W = \cup_{i \in W} \gamma_i$.

Corollary 2.3. *Let π^k be a k-cycle and $K_i \in \pi^k$. Let $W = L \setminus \pi^k$. Given $\varepsilon > 0$, there exist $\rho > 0$ and $N_0 \geq 1$ such that for all $\rho_1 \leq \rho$, $\nu \in \gamma_i \cap \mathcal{M}_1^N(\mathcal{Z})$, and $N \geq N_0$, we have*

$$\exp\{N(\tilde{V}(\pi^k) - \varepsilon)\} \leq E_\nu \hat{\tau}_W \leq \exp\{N(\tilde{V}(\pi^k) + \varepsilon)\}.$$

Corollary 2.4. *Let π_1^k, π_2^k be k-cycles, $\pi_1^k \neq \pi_2^k$, and $K_i \in \pi_1^k$. Let $W = L \setminus \pi_1^k$. Given $\varepsilon > 0$, there exist $\rho > 0$ and $N_0 \geq 1$ such that for all $\rho_1 \leq \rho$, $\nu \in \gamma_i \cap \mathcal{M}_1^N(\mathcal{Z})$, and $N \geq N_0$, we have*

$$\begin{aligned}\exp\{-N(\tilde{V}(\pi_1^k, \pi_2^k) - \tilde{V}(\pi_1^k) + \varepsilon)\} &\leq P_\nu(\mu^N(\hat{\tau}_W) \in \gamma_{\pi_2^k}) \\ &\leq \exp\{-N(\tilde{V}(\pi_1^k, \pi_2^k) - \tilde{V}(\pi_1^k) - \varepsilon)\}.\end{aligned}$$

Remark 2.4. Note that Corollary 2.3 follows from Lemma 2.7 and the fact that $I_i(W) = \tilde{V}(\pi^k)$ (which is shown in [44, Corollary A.4, Appendix]). Corollary 2.4 is a consequence of Lemma 2.5 along with the fact that $\min\{I_{i,j}(W) : i \in \hat{\pi}^k\} = \tilde{V}(\pi^k, \hat{\pi}^k) - \tilde{V}(\pi^k)$ (see [44, Corollary A.6, Appendix]). Similar estimates as in Corollaries 2.3 and 2.4 in the case of small noise diffusion processes have been shown in [44, Corollary 1.10, Part I] and [44, Corollary 1.11, Part I], respectively.

We also need the following lemmas that provide estimates on the probabilities of exit within certain times from given cycles.

Lemma 2.8. *Let π_1^k, π_2^k be k-cycles and let $\pi_1^k \to \pi_2^k$. Then, given $\varepsilon > 0$, there exist $\delta > 0$, $\rho > 0$ and $N_0 \geq 1$ such that for all $\rho_1 \leq \rho$, $\nu \in \gamma_{\pi_1^k} \cap \mathcal{M}_1^N(\mathcal{Z})$, and $N \geq N_0$, we have*

$$P_\nu \left(\bar{\tau}_{\pi_1^k} \leq \exp\{N(\tilde{V}(\pi_1^k) - \delta)\}, \mu^N(\bar{\tau}_{\pi_1^k}) \in \gamma_{\pi_2^k} \right) \geq \exp\{-N\varepsilon\}.$$

Lemma 2.9. *Let π^k be a k-cycle. Then, given $\varepsilon > 0$, there exists $\rho > 0$ such that for all $\rho_1 \leq \rho$, we have*

$$\lim_{N\to\infty} \sup_{\nu \in \gamma_{\pi^k} \cap \mathcal{M}_1^N(\mathcal{Z})} P_\nu \left(\exp\{N(\tilde{V}(\pi^k) - \varepsilon)\} \leq \bar{\tau}_{\pi^k} \leq \exp\{N(\tilde{V}(\pi^k) + \varepsilon)\} \right) = 1.$$

Furthermore, given $\varepsilon > 0$, there exist $\delta > 0$, $\rho > 0$, and $N_0 \geq 1$ such that for all $\rho_1 \leq \rho$, $\nu \in \gamma_{\pi^k} \cap \mathcal{M}_1^N(\mathcal{Z})$, and $N \geq N_0$, we have

$$P_\nu \left(\bar{\tau}_{\pi^k} < \exp\{N(\tilde{V}(\pi^k) - \delta)\} \right) \leq \exp\{-N\varepsilon\}, \text{ and}$$
$$P_\nu \left(\bar{\tau}_{\pi^k} > \exp\{N(\tilde{V}(\pi^k) + \delta)\} \right) \leq \exp\{-N\varepsilon\}.$$

Remark 2.5. Lemma 2.9 can be proved as follows. From Corollary 2.3, we have that the mean exit time from a small neighbourhood of the elements of π_1^k is of the form $\exp\{N\tilde{V}(\pi_1^k)\}$. From Corollary 2.4, we have that, upon exit from a small neighbourhood of the elements of π_1^k, the process μ^N enters a small neighbourhood of the elements of π_2^k is of the form $\exp\{-N(\tilde{V}(\pi_1^k, \pi_2^k) - \tilde{V}(\pi_1^k)\}$. Using these facts, we can proceed via the proof of [37, Chapter 6, Theorem 4.2] to transfer the estimate on the mean of $\bar{\tau}_{\pi_1^k}$ to the estimates on the probability for $\bar{\tau}_{\pi_1^k}$ to lie between $\exp\{N(\tilde{V}(\pi_1^k) - \delta)\}$ and $\exp\{N(\tilde{V}(\pi_1^k) + \delta)\}$. To prove Lemma 2.8, in addition to the above facts, we note that with high probability, the process μ^N enters a small neighbourhood of the elements of π_2^k upon exit from a small neighbourhood of the elements of π_1^k when $\pi_1^k \to \pi_2^k$. Similar estimates as in Lemmas 2.8 and 2.9 in the case of small noise diffusion processes have been shown in [44, Lemma 2.1, Part I] and [44, Lemma 2.2, Part I], respectively.

Lemma 2.10. *Let π^k be a k-cycle and assume that $\tilde{V}(\pi^k) > 0$. Given $\varepsilon > 0$, there exist $\delta > 0$, $\rho > 0$, and $N_0 \geq 1$ such that for all $\rho_1 \leq \rho$, $\nu \in \mathcal{M}_1^N(\mathcal{Z})$, and $N \geq N_0$, we have*

$$P_{0,\nu}(\bar{\tau}_{\pi^k} \leq \exp\{N(\hat{V}(\pi^k) + \delta\}) \leq \exp\{-N(\tilde{V}(\pi^k) - \hat{V}(\pi^k) - \varepsilon)\}.$$

Proof. We proceed via the steps in the proof of [44, Lemma 2.1, Part III]. Let $\pi^{k-1} \in \pi^k$ be a $(k-1)$-cycle such that $\tilde{V}(\pi^{k-1}) = \hat{V}(\pi^k)$. With $\rho_1 > 0$ to be chosen later, for each $n \geq 1$, define the minimum of $\bar{\tau}_{\pi^k}$ and successive entry and exit times from a ρ_1-neighbourhood of π^{k-1} as follows:

$$\begin{aligned}\hat{\theta}_0 &:= \inf\{t > 0 : \mu^N(t) \in [\pi^{k-1}]_{\rho_1}\} \wedge \bar{\tau}_{\pi^k},\\ \bar{\theta}_n &:= \inf\{t > \hat{\theta}_{n-1} : \mu^N(t) \in [L \setminus \pi^{k-1}]_{\rho_1}\} \wedge \bar{\tau}_{\pi^k},\\ \hat{\theta}_{n+1} &:= \inf\{t > \bar{\theta}_n : \mu^N(t) \in [\pi^{k-1}]_{\rho_1}\} \wedge \bar{\tau}_{\pi^k}.\end{aligned}$$

With $\delta > 0$ to be chosen later, using the strong Markov property, for any $\nu \in [\pi^k]_{\rho_1} \cap \mathcal{M}_1^N(\mathcal{Z})$, we have

$$\begin{aligned}&P_\nu(\bar{\tau}_{\pi^k} \leq \exp\{N(\hat{V}(\pi^k) + \delta)\})\\ &\quad = P_\nu(\hat{\theta}_0 = \bar{\tau}_{\pi^k}, \bar{\tau}_{\pi^k} \leq \exp\{N(\hat{V}(\pi^k) + \delta)\})\\ &\qquad + P_\nu\left(\hat{\theta}_0 < \bar{\tau}_{\pi^k}, \bigcup_{n\geq 1}\left\{\bar{\tau}_{\pi^k} = \bar{\theta}_n, \bar{\tau}_{\pi^k} \leq \exp\{N(\hat{V}(\pi^k) + \delta)\}, \bar{\tau}_{\pi^k} \geq \hat{\theta}_{n-1}\right\}\right)\\ &\qquad + P_\nu\left(\hat{\theta}_0 < \bar{\tau}_{\pi^k}, \bigcup_{n\geq 1}\left\{\bar{\tau}_{\pi^k} = \hat{\theta}_n, \bar{\tau}_{\pi^k} \leq \exp\{N(\hat{V}(\pi^k) + \delta)\}, \bar{\tau}_{\pi^k} \geq \bar{\theta}_n\right\}\right). \qquad (2.7)\end{aligned}$$

We now upper bound each of the terms in 2.7. Consider the first term. It can be shown using Corollary 2.4 and [44, Corollary A.6, Appendix] that, there exist $\rho_1 > 0$ and $\delta > 0$ such that for any $\nu \in [\pi^k]_{\rho_1}$ and sufficiently large N, we have

$$P_\nu(\hat{\theta}_0 = \bar{\tau}_{\pi^k}) \leq \exp\{-N(\tilde{V}(\pi^k) - \hat{V}(\pi^k) - \varepsilon)\}.$$

Consider the second term in 2.7. For any $\nu_1 \in [\pi^{k-1}]_{\rho_1} \cap \mathcal{M}_1^N(\mathcal{Z})$, the probability of the unionised event can be upper bounded by

$$\begin{aligned}&P_{\nu_1}\left(\bigcup_{n\geq 1}\left\{\bar{\tau}_{\pi^k} = \bar{\theta}_n, \bar{\tau}_{\pi^k} \leq \exp\{N(\hat{V}(\pi^k) + \delta)\}, \bar{\tau}_{\pi^k} \geq \hat{\theta}_{n-1}\right\}\right)\\ &\quad \leq P_{\nu_1}\left(\bigcup_{n=1}^{M}\left\{\bar{\tau}_{\pi^k} = \bar{\theta}_n, \bar{\tau}_{\pi^k} \leq \exp\{N(\hat{V}(\pi^k) + \delta)\}, \bar{\tau}_{\pi^k} \geq \hat{\theta}_{n-1}\right\}\right)\\ &\qquad + P_{\nu_1}\left(\bigcup_{n\geq M+1}\left\{\bar{\tau}_{\pi^k} = \bar{\theta}_n, \bar{\tau}_{\pi^k} \leq \exp\{N(\hat{V}(\pi^k) + \delta)\}, \bar{\tau}_{\pi^k} \geq \hat{\theta}_{n-1}\right\}\right)\\ &\quad \leq P_{\nu_1}(\bar{\tau}_{\pi^k} = \bar{\theta}_n \text{ and } \bar{\tau}_{\pi^k} \geq \hat{\theta}_{n-1} \text{ for some } n \leq M)\\ &\qquad + P_{\nu_1}(\hat{\theta}_M \leq \exp\{N(\hat{V}(\pi^k) + \delta)\} \text{ and } \hat{\theta}_M \leq \bar{\tau}_{\pi^k})\end{aligned}$$

$$\leq P_{\nu_1}(\hat{\theta}_M = \bar{\tau}_{\pi^k}) + P_{\nu_1}(\hat{\theta}_M \leq \exp\{N(\hat{V}(\pi^k) + \delta)\} \text{ and } \hat{\theta}_M \leq \bar{\tau}_{\pi^k}).$$

Again, the first term above can be bounded by

$$P_{\nu_1}(\hat{\theta}_M \leq \bar{\tau}_{\pi^k}) \leq \exp\{-N(\tilde{V}(\pi^k) - \hat{V}(\pi^k) - \varepsilon)\},$$

for all $\nu_1 \in [\pi^{k-1}]_{\rho_1} \cap \mathcal{M}_1^N(\mathcal{Z})$ and sufficiently large N. The second term can be bounded by $\exp\{-NM\}$ for large enough M, by the same argument used in the proof of [44, Lemma 1.7, Part I]. Choosing M sufficiently large, the above implies that the second term in (2.7) is bounded by $\exp\{-N(\tilde{V}(\pi^k) - \hat{V}(\pi^k) - \varepsilon)\}$. A similar argument gives the same bound for the third term in (2.7). □

2.3.4 LDP for the invariant measure

Using the estimates (2.5) on the transition probabilities of the discrete time Markov chain Z^N, we can study the large deviations of the process μ^N in the stationary regime. Recall that $\wp^N$ denotes the unique invariant probability measure of the process μ^N. Also recall that $G(i)$ is the set of all directed graphs g on L such that (a) every node other than i has exactly one outgoing arrow and (b) there are no closed cycles in g. We state the following result:

Theorem 2.7 ([15, Theorem 2.2]). *Assume (A1), (A2), and (B1). Then, the sequence of invariant measures $\{\wp^N, N \geq 1\}$ satisfies the large deviation principle on $\mathcal{M}_1(\mathcal{Z})$ with rate function s given by*

$$s(\xi) = \min_{1\leq i\leq l}\{W(i) + V(K_i, \xi)\} - \min_{1\leq j\leq l} W(j), \tag{2.8}$$

where

$$W(i) = \min_{g\in G(i)} \sum_{(m,n)\in g} \tilde{V}(m, n).$$

The form of the rate function s in Theorem 2.7 is also related to the form of the invariant measure in the context of Markov chains on finite state spaces whose transition kernels are of the form (2.5); see, for example, [28, Section 1.1]. Also, see [13] for an analogous result in a boundary driven symmetric simple exclusion process, which involves the study of the LDP for the invariant measure in an infinite-dimensional setting. However, our focus is on sharp estimates on the rate of convergence to the invariant measure which is the subject of the next section.

2.3.5 Convergence to the invariant measure

In this section, we prove our first main result on the time required for the convergence of μ^N to its invariant measure.

Let $i_0 \in L$ be such that $\min\{\tilde{V}(g) : g \in G(i_0)\} = \min\{\tilde{V}(g) : g \in G(i), i \in L\}$. We anticipate that K_{i_0} is one of the most stable ω-limit sets (among possibly others) for the dynamics (2.1). This is because Theorem 2.7 tells us that the rate function that governs the LDP for $\{\wp^N, N \geq 1\}$ vanishes on K_{i_0}. Hence, for a large but fixed N, over large time intervals, one expects that there is positive probability (in the exponential scale) for the process μ^N to be in a small neighbourhood of K_{i_0}.

Define

$$\Lambda := \begin{cases} \min\{\tilde{V}(g) : g \in G(\{i\}), i \in L\} & \\ \qquad - \min\{\tilde{V}(g) : g \in G(\{i,j\}), i,j \in L, i \neq j\}, & \text{if } |L| \geq 2, \\ 0, & \text{if } |L| = 1. \end{cases} \tag{2.9}$$

Since we are interested in the case when (2.1) has multiple ω-limit sets, we assume throughout that $\Lambda > 0$. The motivation to define this constant Λ is the following. Since the process μ^N spends most of the time near one of the compact sets K_i, we expect that it mixes well when the discrete time Markov chain Z^N, with transition probabilities of the form $\exp\{-N\tilde{V}(K_i, K_j)\}$ given in (2.5), mixes well. The mixing time of Z^N is determined by the real part of $(1 - \hat{\lambda}_2^N)$, where $\hat{\lambda}_2^N$ is the second largest (in absolute value) eigenvalue of an $l \times l$ transition probability matrix whose $(i,j)th$ entry is given by $\exp\{-N\tilde{V}(K_i, K_j)\}$; it turns out that this scales as $\exp\{-N\Lambda\}$ [37, Chapter 6, Theorem 7.3]. Thus, we expect that, when time is of the order $\exp\{N\Lambda\}$, the process μ^N mixes well.

Let $P_T(\nu, \cdot) = P_\nu(\mu^N(T) \in \cdot)$ denote the transition probability kernel associated with the process μ^N. Note that we suppress the dependence on N for ease of readability. We first show a lower bound for the transition probability $P_T(\nu, K_{i_0})$ of reaching a small neighbourhood of K_{i_0} when T is of the order $\exp\{N(\Lambda - \delta_0)\}$ for some $\delta_0 > 0$.

Theorem 2.8. *Given $\varepsilon > 0$, there exist $\delta_0 > 0$, $\rho > 0$, and $N_0 \geq 1$ such that for all $\rho_1 \leq \rho$, $\nu \in \mathcal{M}_1^N(\mathcal{Z})$, and $N \geq N_0$, we have*

$$P_{T_0}(\nu, \gamma_{i_0}) \geq \exp\{-N\varepsilon\}, \tag{2.10}$$

where $T_0 = \exp\{N(\Lambda - \delta_0)\}$. *Furthermore, there exist* $\nu_0 \in \mathcal{M}_1(\mathcal{Z})$ *and* $\beta > 0$ *such that for all* $\nu \in [\nu_0]_{\rho_1} \cap \mathcal{M}_1^N(\mathcal{Z})$ *and* $N \geq N_0$,

$$P_{T_0}(\nu, \gamma_{i_0}) \leq \exp\{-N\beta\}. \tag{2.11}$$

Proof. We follow the steps in Hwang and Sheu [44, Part I, Theorem 2.3]. With $\rho > 0$ to be chosen later, we first show that (2.10) holds for all $\nu \in \gamma \cap \mathcal{M}_1^N(\mathcal{Z})$. Towards this, let m be the smallest integer such that L_{m+1} is a singleton. For $0 \leq k \leq m$, let $\pi_0^k \in L_k$ be the k-cycle containing i_0. Let $V_k = \max\{\tilde{V}(\pi^k) : \pi^k \subset \pi_0^{k+1}, \pi^k \neq \pi_0^k\}$. Using [44, Lemma A.10, Appendix], we have $\Lambda = \max\{V_k : 0 \leq k \leq m\}$.

Fix $j \in L$ and consider $\nu \in [K_j]_\rho$. Let $\pi_1^m \in L_m$ be such that $K_j \in \pi_1^m$. If $\pi_1^m \neq \pi_0^m$, then we have $\pi_1^m \Rightarrow \pi_0^m$, that is, there exists $\pi_2^m, \pi_3^m, \ldots, \pi_n^m = \pi_0^m, n \leq l$ such that $\pi_1^m \to \pi_2^m \to \pi_3^m \to \cdots \to \pi_n^m = \pi_0^m$. Therefore, with δ to be chosen later, by the strong Markov property, we have

$$\begin{aligned}
&P_\nu(\hat{\tau}_{\pi_0^m} \leq n \exp\{N(V_m - \delta)\}) \\
&\quad \geq E_\nu(\mathbf{1}_{\{\bar{\tau}_{\pi_1^m} \leq \exp\{N(V_m - \delta)\}\}} \cdot \mathbf{1}_{\{\mu^N(\bar{\tau}_{\pi_1^m}) \in \pi_2^m\}} \\
&\qquad \times E_{\mu^N(\bar{\tau}_{\pi_1^m})}(\mathbf{1}_{\{\bar{\tau}_{\pi_2^m} \leq \exp\{N(V_m - \delta)\}\}} \cdot \mathbf{1}_{\{\mu^N(\bar{\tau}_{\pi_2^m}) \in \pi_3^m\}} \\
&\qquad \cdots \times E_{\mu^N(\bar{\tau}_{\pi_{n-2}^m})}(\mathbf{1}_{\{\bar{\tau}_{\pi_{n-1}^m} \leq \exp\{N(V_m - \delta)\}\}} \cdot \mathbf{1}_{\{\mu^N(\bar{\tau}_{\pi_{n-1}^m}) \in \pi_0^m\}}) \\
&\qquad \cdots)).
\end{aligned}$$

Since $V(\pi_i^m) \leq V_m$ for all $1 \leq i \leq n$, the above becomes

$$\begin{aligned}
&P_\nu(\hat{\tau}_{\pi_0^m} \leq n \exp\{N(V_m - \delta)\}) \\
&\quad \geq E_\nu(\mathbf{1}_{\{\bar{\tau}_{\pi_1^m} \leq \exp\{N(\tilde{V}(\pi_1^m) - \delta)\}\}} \cdot \mathbf{1}_{\{\mu^N(\bar{\tau}_{\pi_1^m}) \in \pi_2^m\}} \\
&\qquad \times E_{\mu^N(\bar{\tau}_{\pi_1^m})}(\mathbf{1}_{\{\bar{\tau}_{\pi_2^m} \leq \exp\{N(\tilde{V}(\pi_2^m) - \delta)\}\}} \cdot \mathbf{1}_{\{\mu^N(\bar{\tau}_{\pi_2^m}) \in \pi_3^m\}} \\
&\qquad \cdots \times E_{\mu^N(\bar{\tau}_{\pi_{n-2}^m})}(\mathbf{1}_{\{\bar{\tau}_{\pi_{n-1}^m} \leq \exp\{N(\tilde{V}(\pi_{n-1}^m) - \delta)\}\}} \cdot \mathbf{1}_{\{\mu^N(\bar{\tau}_{\pi_{n-1}^m}) \in \pi_0^m\}}) \\
&\qquad \cdots)).
\end{aligned}$$

By Lemma 2.8, there exist $\rho > 0$, $\delta > 0$ and $N_0 \geq 1$ such that each of the above probabilities is at least $\exp\{-N\varepsilon/l\}$ for sufficiently large N, i.e. we have

$$P_\nu(\hat{\tau}_{\pi_0^m} \leq n \exp\{N(V_m - \delta))\}) \geq \exp\{-Nn\varepsilon/l\} \geq \exp\{-N\varepsilon\},$$

On the other hand, if K_j is such that $K_j \in \pi_0^m$, the above holds trivially. Therefore, there exist $\delta_1 > 0$ and $N_1 \geq 1$ such that for all $\nu \in \gamma \cap \mathcal{M}_1^N(\mathcal{Z})$ and $N \geq N_1$, we have

$$P_\nu(\hat{\tau}_{\pi_0^m} \leq \exp\{N(V_m - \delta_1)\}) \geq \exp\{-N\varepsilon\}.$$

We now use the above bound to show (2.10). Let $T = \exp\{N(\Lambda - \delta_1)\}, T_m = \exp\{N(V_m - \delta_1)\}$ and $T_{m-1} = \exp\{N(V_{m-1} - \delta_1)\}$. Then, for any $\nu \in \gamma \cap \mathcal{M}_1^N(\mathcal{Z})$ and $N \geq N_1$, we have

$$\begin{aligned}
P_\nu(\mu^N(T) \in \gamma_{i_0}) &\geq E_\nu(\mathbf{1}_{\{\hat{\tau}_{\pi_0^m} \leq T_m\}} \cdot \mathbf{1}_{\{\mu^N(T - \hat{\tau}_{\pi_0^m}) \in \gamma_{i_0}\}}) \\
&= E_\nu(\mathbf{1}_{\{\hat{\tau}_{\pi_0^m} \leq T_m\}} \cdot E_{\mu^N(\hat{\tau}_{\pi_0^m})}(\mathbf{1}_{\{\mu^N(T - \hat{\tau}_{\pi_0^m}) \in \gamma_{i_0}\}})) \\
&\geq \inf_{\substack{\nu' \in [\pi_0^m]_\rho \cap \mathcal{M}_1^N(\mathcal{Z}) \\ T - T_m \leq t \leq T}} P_{\nu'}(\mu^N(t) \in \gamma_{i_0}) P_\nu(\hat{\tau}_{\pi_0^m} \leq T_m) \\
&\geq \inf_{\substack{\nu' \in [\pi_0^m]_\rho \cap \mathcal{M}_1^N(\mathcal{Z}) \\ T - T_m \leq t \leq T}} P_{\nu'}(\mu^N(t) \in \gamma_{i_0}) \exp\{-N\varepsilon\}, \qquad (2.12)
\end{aligned}$$

where the second equality follows from the strong Markov property. To get a lower bound for the above infimum, fix $\nu \in [\pi_0^m]_\rho \cap \mathcal{M}_1^N(\mathcal{Z})$ and $T - T_m \leq t \leq T$. Define the stopping time $\theta \coloneqq \inf\{s > t - T_{m-1} : \mu^N(s) \in [\pi_0^m]_\rho\}$. Then, for a large T^* (not depending on N) to be chosen later, we have

$$\begin{aligned}
&P_\nu(\mu^N(t) \in \gamma_{i_0}) \\
&\quad \geq E_\nu(\mathbf{1}_{\{\theta \leq t - T_{m-1} + T^*, \bar{\tau}_{\pi_0^m} > T\}} \cdot E_{\mu^N(\theta)}(\mathbf{1}_{\{\mu^N(t - \theta) \in \gamma_{i_0}\}})) \\
&\quad \geq P_\nu(\theta \leq t - T_{m-1} + T^*, \bar{\tau}_{\pi_0^m} > T) \inf_{\substack{\nu' \in [\pi_0^m]_\rho \cap \mathcal{M}_1^N(\mathcal{Z}) \\ T_{m-1} - T^* \leq t \leq T_{m-1}}} P_{\nu'}(\mu^N(t) \in \gamma_{i_0}). \qquad (2.13)
\end{aligned}$$

Note that

$$P_\nu(\theta \leq t - T_{m-1} + T^*, \bar{\tau}_{\pi_0^m} > T) = P_\nu(\bar{\tau}_{\pi_0^m} > T) - P_\nu(\theta > t - T_{m-1} + T^*, \bar{\tau}_{\pi_0^m} > T).$$

By Lemma 2.9, since $\Lambda \leq \tilde{V}(\pi_0^m)$, we have

$$P_\nu(\bar{\tau}_{\pi_0^m} > T) \geq P_\nu(\bar{\tau}_{\pi_0^m} > \exp\{N(\tilde{V}(\pi_0^m) - \delta)\}) \to 1$$

as $N \to \infty$. For the second term, note that

$$P_\nu(\theta > t - T_{m-1} + T^*, \bar{\tau}_{\pi_0^m} > T)$$

$$= P_\nu(\mu^N(s) \notin [\pi_0^m]_\rho \text{ for all } t - T_{m-1} \le s \le t - T_{m-1} + T^*, \bar{\tau}_{\pi_0^m} > T)$$
$$= P_\nu(\mu^N(s) \notin \gamma \text{ for all } t - T_{m-1} \le s \le t - T_{m-1} + T^*, \bar{\tau}_{\pi_0^m} > T)$$
$$\le P_\nu(\mu^N(s) \notin \gamma \text{ for all } t - T_{m-1} \le s \le t - T_{m-1} + T^*).$$

The second equality follows since $\mu^N(s) \notin [\pi_0^m]_\rho$ and $\bar{\tau}_{\pi_0^m} > T$ implies that we have exited $[\pi_0^m]_\rho$ and we have not yet entered a neighbourhood of any other attractor, which is the same as saying $\mu^N(t) \notin \gamma$ and $\bar{\tau}_{\pi_0^m} > T$. By the Markov property, the above probability equals

$$E_\nu\left(E_{\mu^N(t-T_{m-1})}(\mathbf{1}_{\{\mu^N(s)\notin\gamma \text{ for all } s\in[t-T_{m-1},t-T_{m-1}+T^*]\}})\right) \le \sup_{\nu'\in F} P_{\nu'}(\tau_F \ge T^*),$$

where $F = \mathcal{M}_1(\mathcal{Z}) \setminus \gamma$. By Lemma 2.3, T^* can be chosen large enough (not depending on N) that the above probability is at most $1/2$. Therefore, (2.13) becomes

$$\inf_{\substack{\nu\in[\pi_0^m]_\rho\cap\mathcal{M}_1^N(\mathcal{Z})\\ T-T_m\le t\le T}} P_\nu(\mu^N(t) \in \gamma_{i_0}) \ge \frac{1}{2} \inf_{\substack{\nu'\in[\pi_0^m]_\rho\cap\mathcal{M}_1^N(\mathcal{Z})\\ T_{m-1}-T^*\le t\le T_{m-1}}} P_{\nu'}(\mu^N(t) \in \gamma_{i_0}),$$

and (2.12) becomes

$$P_\nu(\mu^N(T) \in \gamma_{i_0}) \ge \frac{1}{2}\exp\{-N\varepsilon\} \inf_{\substack{\nu'\in[\pi_0^m]_\rho\cap\mathcal{M}_1^N(\mathcal{Z})\\ T_{m-1}-T^*\le t\le T_{m-1}}} P_{\nu'}(\mu^N(t) \in \gamma_{i_0}),$$

for sufficiently large N and $\nu \in \gamma \cap \mathcal{M}_1^N(\mathcal{Z})$. Repeating the above argument m times, we see that there exists $N_2 \ge 1$ such that for all $\nu \in \gamma$ and $N \ge N_2$, we have

$$\begin{aligned} P_\nu(\mu^N(T) \in \gamma_{i_0}) &\ge \left(\frac{1}{2}\right)^m \exp\{-Nm\varepsilon\} \inf_{\substack{\nu'\in[\pi_0^1]_\rho\cap\mathcal{M}_1^N(\mathcal{Z})\\ T_0-T^*\le t\le T_0}} P_{\nu'}(\mu^N(t) \in \gamma_{i_0}) \\ &\ge \left(\frac{1}{2}\right)^m \exp\{-N(m+1)\varepsilon\} \inf_{\substack{\nu'\in[K_0]_\rho\cap\mathcal{M}_1^N(\mathcal{Z})\\ T_0-T^*\le t\le T_0}} P_{\nu'}(\mu^N(t) \in \gamma_{i_0}) \\ &\ge \left(\frac{1}{2}\right)^{m+1} \exp\{-N(m+1)\varepsilon\}, \end{aligned}$$

where $T_0 = \exp\{N(V_0 - m\delta)\}$. Thus, we conclude that there is $N_3 \ge 1$, $\delta_3 > 0$ and $\rho > 0$ such that for all $\nu \in \gamma \cap \mathcal{M}_1^N(\mathcal{Z})$ and $N \ge N_3$, we have

$$P_\nu(\mu^N(T) \in \gamma_{i_0}) \ge \exp\{-N(m+3)\varepsilon\},$$

where $T = \exp\{N(\Lambda - \delta_3)\}$. This establishes (2.10) for all $\nu \in \gamma \cap \mathcal{M}_1^N(\mathcal{Z})$. For any $\nu \in \mathcal{M}_1^N(\mathcal{Z}) \setminus \gamma$, from Lemma 2.3, there exists T' large enough and $N_4 \geq N_3$ such that $P_\nu(\tau_{\mathcal{M}_1(\mathcal{Z})\setminus\gamma} \leq T') \geq \frac{1}{2}$ for all $N \geq N_4$. Therefore, we have

$$\begin{aligned} P_\nu(\mu^N(T) \in \gamma_{i_0}) &\geq E_\nu(\mathbf{1}_{\{\tau_{\mathcal{M}_1^N(\mathcal{Z})\setminus\gamma} \leq T'\}} \cdot E_{\mu^N(\tau_F)}(\mathbf{1}_{\{\mu^N(T-T') \in \gamma_{i_0}\}})) \\ &\geq \frac{1}{2} \inf_{\nu' \in \gamma} P_{\nu'}(\mu^N(T - T') \in \gamma_{i_0}) \\ &\geq \frac{1}{2} \exp\{-N(m+3)\varepsilon\}. \end{aligned}$$

Thus, we have established (2.10) for any $\nu \in \mathcal{M}_1^N(\mathcal{Z})$.

We now turn to (2.11). Since $\Lambda = \max\{V_k, 0 \leq k \leq m\}$, there exists a k such that $V_k = \Lambda$. From the definition of V_k, we see that there exists $\pi^k \in L_k$ such that

$$\tilde{V}(\pi^k) = \Lambda, \pi^k \subset \pi_0^{k+1}, \text{ and } \pi^k \neq \pi_0^k.$$

where π_0^{k+1} is the $(k+1)$-cycle that contain K_{i_0}. Therefore, Lemma 2.9 implies that, for some $\beta > 0$, for some $\delta_4 < \delta_3$ and an appropriately chosen $\rho > 0$, with $T = \exp\{N(\Lambda - \delta_3)\} = \exp\{N(\tilde{V}(\pi^k) - \delta_3)\}$, we have

$$P_\nu(\mu^N(T) \in \gamma_{i_0}) \leq P_\nu(\bar{\tau}_{\pi^k} \leq T) \leq \exp\{-N\beta\},$$

for any $\nu \in [\pi^k]_\rho \cap \mathcal{M}_1^N(\mathcal{Z})$ and sufficiently large N. This completes the proof of the theorem. □

The above theorem immediately gives a lower bound on $P_T(\nu, \xi)$ for any ξ in a small neighbourhood of K_{i_0}, over time durations of order $\exp\{N(\Lambda - \delta)\}$ for some $\delta > 0$. Let us make this precise.

Corollary 2.5. *Under the conditions of Theorem 2.8, for all $\nu \in \mathcal{M}_1^N(\mathcal{Z})$, $\xi \in \gamma_{i_0} \cap \mathcal{M}_1^N(\mathcal{Z})$ and N sufficiently large, we have*

$$P_{T_0}(\nu, \xi) \geq \exp\{-2N\varepsilon\}.$$

Proof. Given $\varepsilon > 0$, let ρ, N_0 and T_0 be as in the statement of Theorem 2.8. Choose t large enough (not depending on N) and $\rho' < \rho$ such that for all $\rho_1 \leq \rho'$ we have $S_t(\nu_1|\nu_2) \leq \varepsilon/2$ for all $\nu_1, \nu_2 \in \gamma_{i_0}$. This is possible by the joint continuity of the rate function $S_t(\cdot|\cdot)$ and the fact that $V(\nu_1, \nu_2) = 0$ whenever $\nu_1, \nu_2 \in K_{i_0}$. Therefore, using the large deviation lower bound, there exists $N_2 \geq N_1$ such that

$$P_t(\nu_1, \nu_2) \geq \exp\{-N(S_t(\nu_2|\nu_1) + \varepsilon/2)\} \geq \exp\{-N\varepsilon\},$$

for all $\nu_1, \nu_2 \in \gamma_{i_0} \cap \mathcal{M}_1^N(\mathcal{Z})$ and $N \geq N_2$. Therefore, by Theorem 2.8, for $\nu \in \mathcal{M}_1^N(\mathcal{Z}), \xi \in \gamma_{i_0} \cap \mathcal{M}_1^N(\mathcal{Z})$ and $N \geq N_2$, we have

$$\begin{aligned} P_{T_0}(\nu, \xi) &= \sum_{\nu_2 \in \gamma_{i_0} \cap \mathcal{M}_1^N(\mathcal{Z})} P_{T_0 - t}(\nu_1, \nu_2) P_t(\nu_2, \xi) \\ &\geq P_{T_0 - t}(\nu_1, \gamma_{i_0}) \inf_{\nu_2 \in \gamma_{i_0} \cap \mathcal{M}_1^N(\mathcal{Z})} P_t(\nu_2, \xi) \\ &\geq \exp\{-2N\varepsilon\}. \end{aligned}$$

□

2.3.5.1 Proofs of Theorem 2.1 and Theorem 2.2

We now prove our first main result (Theorem 2.1) on the convergence of μ^N to the invariant measure and its converse Theorem 2.2. Theorem 2.1 together with Theorem 2.2 shows that the constant Λ is sharp (in the exponential scale) for the time required for μ^N to equilibrate.

Define $\tilde{L}_0 \coloneqq \{i \in L : s(K_i) = 0\}$, i.e, $\tilde{L}_0$ denotes the set of minimisers of the rate function s (see (2.8)). Let $B(\mathcal{M}_1(\mathcal{Z}))$ denotes the space of bounded Borel-measurable functions on $\mathcal{M}_1(\mathcal{Z})$.

Proof of Theorem 2.1. We follow the steps in Hwang and Sheu [44, Part I, Theorem 2.5]. Let $\varepsilon > 0$, and let $T_0, \delta_0, \rho, \rho_1$, and $N_0 \geq 1$ be as in the statement of Theorem 2.8. Note that, for any $\nu \in \mathcal{M}_1^N(\mathcal{Z})$, $\xi \notin [\tilde{L}_0]_{\rho_1}$ and for some fixed $t > 0$,

$$\begin{aligned} P_{T_0}(\nu, \xi) &= \sum_{\nu' \in [K_{i_0}]} P_{T_0 - t}(\nu, \nu') P_t(\nu', \xi) \\ &\geq \exp\{-2N\varepsilon\} \inf_{\nu' \in [K_{i_0}]} P_t(\nu', \xi) \\ &\geq \exp\{-2N\varepsilon\} \exp\{-N \sup_{\nu' \in [K_{i_0}]} S_t(\xi|\nu')\} \end{aligned}$$

where the first inequality follows from Corollary 2.5 and the second from the uniform LDP (Corollary 2.1). Hence, we can find a function $U : \mathcal{M}_1(\mathcal{Z}) \to [0, \infty)$ such that $U(\xi) = 0$ for

$\xi \in [\tilde{L}_0]_{\rho_1}$ and

$$P_{T_0}(\nu, \xi) \geq c_N \exp\{-NU(\xi)\} \tag{2.14}$$

holds for all $\nu \in \mathcal{M}_1^N(\mathcal{Z})$, $\xi \notin [\tilde{L}_0]_{\rho_1}$ and sufficiently large N; here c_N is such that

$$\pi_N(\xi) = c_N \exp\{-NU(\xi)\}$$

is a probability measure on $\mathcal{M}_1^N(\mathcal{Z})$. Define $Q_{T_0}(\nu, \cdot) := P_{T_0}(\nu, \cdot)/\pi_N(\cdot)$. Note that $c_N \to 0$ exponentially fast as $N \to \infty$. Indeed, since $U(\xi) = 0$ for all $\xi \in [\tilde{L}_0]_{\rho_1}$, each of these points yield $\pi_N(\xi) = c_N$. Since the number of points in $[\tilde{L}_0]_{\rho_1} \cap \mathcal{M}_1^N(\mathcal{Z})$ is exponential in N and since π_N is a probability measure, we see that c_N must decay exponentially fast in N. We have, for any $\nu_1, \nu_2 \in \mathcal{M}_1^N(\mathcal{Z})$ and sufficiently large N,

$$\begin{aligned}
&E_{\nu_1}(f(\mu^N(T_0))) - E_{\nu_2}(f(\mu^N(T_0))) \\
&\quad = \sum_{\xi \in \mathcal{M}_1^N(\mathcal{Z})} P_{T_0}(\nu_1, \xi) f(\xi) - \sum_{\xi \in \mathcal{M}_1^N(\mathcal{Z})} P_{T_0}(\nu_2, \xi) f(\xi) \\
&\quad = \sum_{\xi \in \mathcal{M}_1^N(\mathcal{Z})} Q_{T_0}(\nu_1, \xi) f(\xi) \pi_N(\xi) - \sum_{\xi \in \mathcal{M}_1^N(\mathcal{Z})} Q_{T_0}(\nu_2, \xi) f(\xi) \pi_N(\xi) \\
&\quad = \sum_{\xi \in \mathcal{M}_1^N(\mathcal{Z})} (Q_{T_0}(\nu_1, \xi) - \exp\{-2N\varepsilon\}) f(\xi) \pi_N(\xi) \\
&\qquad - \sum_{\xi \in \mathcal{M}_1^N(\mathcal{Z})} (Q_{T_0}(\nu_2, \xi) - \exp\{-2N\varepsilon\}) f(\xi) \pi_N(\xi) \\
&\quad \leq (1 - \exp\{-2N\varepsilon\})(\sup_\xi f(\xi) - \inf_\xi f(\xi)),
\end{aligned}$$

where the last inequality follows from (2.14) and the fact that $Q_{T_0}(\cdot, \cdot) \geq 1$. Therefore, we have that

$$\sup_{\nu_1, \nu_2} |E_{\nu_1}(f(\mu^N(T_0))) - E_{\nu_2}(f(\mu^N(T_0)))| \leq (1 - \exp\{-2N\varepsilon\}) \|f\|_\infty.$$

Continuing this procedure k times, and by using the Markov property, we get

$$\sup_{\nu_1, \nu_2} |E_{\nu_1}(f(\mu^N(kT_0))) - E_{\nu_2}(f(\mu^N(kT_0)))| \leq (1 - \exp\{-2N\varepsilon\})^k \|f\|_\infty,$$

and hence, we have

$$\sup_{\nu} |E_\nu(f(\mu^N(kT_0))) - \langle f, \wp^N \rangle| \le (1 - \exp\{-2N\varepsilon\})^k \|f\|_\infty.$$

Choose $k = \exp\{N(\delta_0 + \delta)\}$, then we have $kT_0 = \exp\{N(\Lambda + \delta)\}$ and the above becomes

$$\sup_{\nu} |E_\nu(f(\mu^N(kT_0))) - \langle f, \wp^N \rangle| \le \exp\{-\exp(N(-2\varepsilon + \delta_0 + \delta))\}.$$

We can choose ε small enough such that the quantity $-2\varepsilon + \delta > 0$, and hence for some $\varepsilon' > 0$, we have

$$\sup_{\nu} |E_\nu(f(\mu^N(T))) - \langle f, \wp^N \rangle| \le \exp\{-\exp(N\varepsilon')\},$$

for sufficiently large N, where $T = \exp\{N(\Lambda + \delta)\}$. This establishes the result. □

Proof of Theorem 2.2. This is a direct consequence of (2.11) established in Theorem 2.8. □

2.4 Asymptotics of the second largest eigenvalue for reversible processes

In this section, our goal is to understand the convergence rate of μ^N to its invariant measure for a fixed N. For this purpose, we shall assume that the Markov process μ^N is reversible. That is, the operator L^N is self-adjoint in $L^2(\wp^N)$ and it admits a spectral expansion; let $0 = \lambda_1^N > -\lambda_2^N \ge -\lambda_3^N \ldots$ denote its eigenvalues in the decreasing order, and let $u_1^N \equiv 1, u_2^N, u_3^N, \ldots$ denote their corresponding eigenfunctions. Without loss of generality, we assume that the eigvenfunctions are orthonormal, i.e., $(u_i^N, u_i^N) = 1$ for each i and $(u_i^N, u_j^N) = 0$ for each $i \neq j$, where $(\cdot, \cdot)$ denotes the inner product in $L^2(\wp^N)$. The spectral expansion [5, Section 1.7.2] enables us to write, for any $f \in B(\mathcal{M}_1(\mathcal{Z}))$,

$$E_\nu f(\mu^N(t)) = \langle f, \wp^N \rangle + \sum_{k \ge 2} e^{-t\lambda_k^N} (f, u_k^N) u_k^N(\nu), \tag{2.15}$$

Therefore, the convergence rate of $E_\nu f(\mu^N(t))$ to its stationary value $\langle f, \wp^N \rangle$ is determined by the leading term in the above sum, which is the second largest eigenvalue λ_2^N. Hence, to understand convergence of μ^N to its invariant measure, we study the asymptotics of the second largest eigenvalue λ_2^N.

We first need the following lemma that estimates the probability that the process μ^N is outside a small neighbourhood of the set $\cup_{i=1}^l K_i$.

Lemma 2.11. *Fix $\rho_1 > 0$ and let B be the ρ_1-neighbourhood of $\cup_{i\in L} K_i$. Given $\varepsilon > 0$, there exist $\delta > 0$ and $N_0 \geq 1$ such that for each $\nu \in \mathcal{M}_1^N(\mathcal{Z})$ and $N \geq N_0$, we have*

$$P_\nu\left(\mu^N(T) \in \mathcal{M}_1^N(\mathcal{Z}) \setminus B\right) \leq \exp\{-N\delta\},$$

where $T = \exp\{N(\Lambda + \varepsilon)\}$.

This result can be proved using Theorem 2.1 which deals with the convergence to the invariant measure and Theorem 2.7 which addresses large deviations of the invariant measure $\{\wp^N, N \geq 1\}$. Indeed, from Theorem 2.7, since the function s in (2.8) is strictly positive outside $\cup_{i=1}^l K_i$, the probability of the complement of a small neighbourhood of $\cup_{i=1}^l K_i$ under $\wp^N$ decays exponentially in N. But when T is of the order $\exp\{N(\Lambda + \varepsilon)\}$, from Theorem 2.1, the law of the random variable $\mu^N(T)$ is not far from $\wp^N$. Therefore, the probability that $\mu^N(T)$ lies outside a small neighbourhood of $\cup_{i=1}^l K_i$ decays exponentially in N.

We are now ready to prove our next main result (Theorem 2.3) on the asymptotics of the second largest eigenvalue λ_2^N.

Proof of Theorem 2.3. (Lower bound): Suppose that there exists a subsequence $\{N_k, k \geq 1\}$ such that

$$\log \lambda_2^{N_k} < -N_k(\Lambda + \varepsilon) \tag{2.16}$$

for some $\varepsilon > 0$. We will show that this contradicts $\int (u_2^{N_k}(\nu))^2 \wp^N(d\nu) = 1$ for sufficiently large k. Fix $\rho > 0$ and define $B := \cup_{i=1}^l [K_i]_\rho$. Then, using the lower semicontinuity of the rate function $S_t(\cdot|\cdot)$ and Corollary 2.1 on uniform LDP, we see that for sufficiently large t, there exists $\delta_1 > 0$ such that $\inf\{S_t(\xi|\nu) : \xi, \nu \in B^c\} = \delta_1 > 0$. Therefore, for any $\nu \in B^c \cap \mathcal{M}_1^N(\mathcal{Z})$ and any $\delta_2 > 0$, there exists $N_0 \geq 1$ such that for all $N \geq N_0$,

$$P_\nu(\mu^N(t) = \nu) \leq \exp\{-N(S_t(\nu|\nu) - \delta_2)\} \leq \exp\{-N(\delta_1 + \delta_2)\}.$$

On the other hand, (2.15) implies that,

$$\begin{aligned} P_\nu(\mu^N(t) = \nu) &= E_\nu(\mathbf{1}_{\{\nu\}}(\mu^N(t))) \\ &\geq e^{-\lambda_2^N t}(u_2^N(\nu))^2 \wp^N(\nu), \end{aligned}$$

so that

$$\int_{B^c} |u_2^N|^2 \wp^N(d\nu) \le \exp\{-N(\delta_1+\delta_2)\} \tag{2.17}$$

for all $N \ge N_0$. To bound the integral over B, by Theorem 2.1, with $T = \exp\{N(\Lambda+\varepsilon/2)\}$, there exist $\delta_3 > 0$ and $N_1 \ge N_0$ such that for all $N \ge N_1$,

$$\left|E_\nu f(\mu^N(T)) - \langle f, \wp^N\rangle\right| \le \|f\|_\infty \exp\{-\exp(N\delta_3)\},$$

for any $f \in B(\mathcal{M}_1(\mathcal{Z}))$. On the other hand, from (2.15), for any $\nu \in B \cap \mathcal{M}_1^N(\mathcal{Z})$, with $f = \mathbf{1}_{\{\nu\}}$, we have

$$\begin{aligned}\left|E_\nu f(\mu^N(T)) - \langle f, \wp^N\rangle\right| &= \left|\sum_{i\ge 2} \exp\{-\lambda_i^N T\}(f, u_i^N(\nu))u_i^N(\nu)\right| \\ &\ge \exp\{-\lambda_2^N T\}(u_2^N(\nu))^2\wp^N(\nu),\end{aligned}$$

so that, by our assumption (2.16), there exists a $k_0 \ge 1$ such that

$$\begin{aligned}u_2^{N_k}(\nu))^2\wp^{N_k}(\nu) &\le \exp\{\lambda_2^{N_k}T\}\exp\{-\exp(N_k\delta_3)\} \\ &\le \exp\{2\exp(-N_k(\Lambda+\varepsilon))\exp(N_k(\Lambda+\varepsilon/2))\}\exp\{-N_k\delta_3\}\end{aligned}$$

for all $k \ge k_0$. Since $|\mathcal{M}_1^{N_k}(\mathcal{Z})| \le (N_k+1)^{|\mathcal{Z}|}$ for all k, the above implies that, for some $\delta_4 > 0$,

$$\int_B (u_2^{N_k}(\nu))^2\wp^{N_k}(d\nu) \le \exp\{-N_k\delta_4\} \tag{2.18}$$

for all $k \ge k_0$. Therefore, (2.17) and (2.18) implies that, for some $\delta > 0$,

$$\int_{\mathcal{M}_1(\mathcal{Z})} (u_2^{N_k}(\nu))^2\wp^{N_k}(d\nu) \le \exp\{-N_k\delta\}$$

for all sufficiently large k, which is a contradiction to $\int (u_2^{N_k}(\nu))^2\wp^{N_k}(d\nu) = 1$ for all sufficiently large k.

(Upper bound): Suppose that there exists a subsequence $\{N_k, k \ge 1\}$ such that $\log\lambda_2^N > N_k(-\Lambda+\varepsilon)$ for some $\varepsilon > 0$. Let $\nu_0, \delta_0 < \varepsilon/2, \rho, N_0$ be as in Theorem 2.8. Then, with

$f(\nu) = \mathbf{1}_{\{[K_{i_0}]_{\rho/2}\}}(\nu)$ and $T = \exp\{N(\Lambda - \delta_0/2)\}$, (2.11) implies that

$$E_\nu f(\mu^N(T)) = P_\nu(\mu^N(T) \in [K_{i_0}]_{\rho/2}) \le \exp\{-N\beta\}$$

for all $N \ge N_0$ and $\nu \in [\nu_0]_{\rho/2} \cap \mathcal{M}_1^N(\mathcal{Z})$. Also, by Theorem 2.7, for any $\delta > 0$, there exists $N_1 \ge N_0$ such that for all $N \ge N_1$, we have

$$\langle f, \wp^N \rangle = \wp^N([K_{i_0}]_{\rho/2}) \ge \exp\{-N\delta\}.$$

This is possible since $\inf_{\xi \in [K_{i_0}]_{\rho/2}} s(\xi) = 0$. Therefore, for all $N \ge N_1$,

$$\begin{aligned}
\int_{\mathcal{M}_1(\mathcal{Z})} &|E_\nu(f(\mu^N(T))) - \langle f, \wp^N \rangle|^2 \wp^N(d\nu) \\
&\ge \int_{[\nu_0]_{\rho/2}} \left| E_\nu(f(\mu^N(T))) - \langle f, \wp^N \rangle \right|^2 \wp^N(d\nu) \\
&\ge \wp^N([\nu_0]_{\rho/2})(\exp\{-N\beta\} - \exp\{-N\delta\}) \\
&\ge \wp^N([\nu_0]_{\rho/2}) \exp\{-N\delta_1\}, \text{ for some } \delta_1 > 0 \\
&\ge \exp\{-N\delta_2\}, \text{ for some } \delta_2 > 0,
\end{aligned}$$

where the last inequality follows by Theorem 2.7. On the other hand, for any function f with $\int |f|^2 d\wp^N \le 1$, we have

$$\begin{aligned}
\int_{\mathcal{M}_1(\mathcal{Z})} &|E_\nu(f(\mu^N(T))) - \langle f, \wp^N \rangle|^2 \wp^N(d\nu) \\
&= \int_{\mathcal{M}_1(\mathcal{Z})} \sum_{k \ge 2} e^{-2\lambda_k^N T} (f, u_k^N)^2 u_k^N(\nu)^2 \wp^N(d\nu) \\
&\le \exp\{-2\lambda_2^N T\} \int_{\mathcal{M}_1(\mathcal{Z})} |f|^2 d\wp^N \\
&\le \exp\{-2\lambda_2^N T\}.
\end{aligned}$$

Therefore, we have $\exp\{-2\lambda_2^N T\} \ge \exp\{-N\delta_2\}$ whenever $N \ge N_1$. By our assumption, we see that

$$\exp\{-2\exp(-N_k(\Lambda - \varepsilon))\exp(N_k(\Lambda - \delta_0))\} \ge \exp\{-N_k\delta_1\}$$

for sufficiently large k, which is a contradiction since $\delta_0 < \varepsilon/2$. $\square$

Using the above theorem, we see that, if $\Lambda > 0$, then as N becomes large, it takes longer for the process μ^N to be close to its invariant measure. This particularly means that metastable states reduce the rates of convergence of μ^N to its invariant measure. On the other hand, if there is a unique global attractor of the limiting McKean-Vlasov equation (2.1), then we see that $\Lambda = 0$, and convergence rate of μ^N to its invariant measure does not suffer from such a slowing down phenomenon.

Note that the spectral expansion in (2.15) is crucial in the proof of Theorem 2.3 to be able to use the results on the large time behaviour of μ^N established in Section 2.3 to obtain the asymptotics of λ_2^N. The main purpose of Theorem 2.3 is to demonstrate that, in the reversible case, the asymptotics of λ_2^N can be easily obtained as an application of the study of the large time behaviour of μ^N. Even in the non-reversible case, one can obtain asymptotics of the real part of λ_2^N via other approaches; see, for example, [89], where the author obtains the asymptotics of the real part of the second largest eigenvalue of the generator corresponding to a small noise diffusion process via examining eigenvalues of a discrete time chain (with transition probabilities of the form appearing in (2.5)) and transferring them to the operator. Further, the asymptotics of all the eigenvalues of an $l \times l$ transition probability matrix whose $(i,j)th$ entry, for $i \neq j$, is given by $\exp\{-N\tilde{V}(K_i, K_j)\}$ can also be obtained; the real part of $(1 - \hat{\lambda}_k^N)$ (where $\hat{\lambda}_k^N$ is the kth eigenvalue of the matrix, $2 \leq k \leq l$) decays exponentially in N where the exponent is given by a quantity analogous to Λ in (2.9) in which the first minimum is taken over all graphs in $G(W)$ with $|W| = k - 1$ and the second minimum is taken over all graphs in $G(W)$ with $|W| = k$, see Freidlin and Wentzell [37, Chapter 6, Theorem 7.3]. However, it is not clear how to transfer these asymptotics to the eigenvalues of L^N using the large time behaviour of μ^N, a question that we leave for the future. This question is also related to the behaviour of μ^N over times of the order of the inverse of these eigenvalues. For reversible Markov chains with a small parameter, such questions have been studied by Miclo [61, 62]. In particular, it would be interesting to investigate the asymptotics of λ_3^N, since the convergence rate of μ^N to $\wp^N$ depends on whether λ_3^N decays as $\exp\{-N\Lambda\}$ or $\lambda_3^N \gg \exp\{-N\Lambda\}$.

Example 2.2. We provide a simplified Curie-Weiss model of magnetisation for which L^N is reversible with respect to $\wp^N$ for all $N \geq 1$ [25]. Let $\mathcal{Z} = \{-1, +1\}$. The states represent the direction of magnetisation of the particles. For each $N \geq 1$, consider the following probability measure on $\mathcal{Z}^N$:

$$\pi_N(\mathbf{z}^N) = \frac{1}{C_N} \exp\left\{ N \left(\frac{1}{N} \sum_{n=1}^{N} z_n \right)^2 \right\}, \ \mathbf{z}^N = (z_1, z_2, \ldots, z_N) \in \mathcal{Z}^N, \tag{2.19}$$

where C_N is a normalisation constant. Given $\xi \in \mathcal{M}_1(\mathcal{Z})$, define the total magnetisation by $\xi_{\text{tot}} = \xi(+1) - \xi(-1)$. Define the transition rates

$$\lambda_{z,(-z)}(\xi) = \exp\{-2z\xi_{\text{tot}}\},\ z \in \mathcal{Z},\ \xi \in \mathcal{M}_1(\mathcal{Z}).$$

It is straightforward to verify that the Markov process $(X_1^N, \ldots, X_N^N)$ that describes the joint evolution of the states of all the particles is reversible with respect to its invariant measure π_N in (2.19). That is, for every $\mathbf{z}^N, \tilde{\mathbf{z}}^N \in \mathcal{Z}^N$ that differ on the nth component, we have (recall that $\overline{\mathbf{z}^N} = \frac{1}{N}\sum_{n=1}^N \delta_{z_n}$)

$$\pi_N(\mathbf{z}^N)\lambda_{z_n,(-z_n)}(\overline{\mathbf{z}^N}) = \pi_N(\tilde{\mathbf{z}}^N)\lambda_{(-z_n),z_n}(\overline{\tilde{\mathbf{z}}^N}). \tag{2.20}$$

From the reversibility of $(X_1^N, \ldots, X_N^N)$, noting that $\wp^N(\xi)$ is the sum of $\pi^N(\mathbf{z}^N)$ over all $\mathbf{z}^N$ such that $\overline{\mathbf{z}^N} = \xi$, it is straightforward to check that μ^N is reversible. For completeness, we show the reversibility of μ^N by checking the detailed balance condition. Towards this, we first note that for any $\xi, \tilde{\xi} \in \mathcal{M}_1^N(\mathcal{Z})$ such that $\xi(z) = \tilde{\xi}(z) + 1/N$ for some $z \in \mathcal{Z}$ (which ensures that $\xi(z) > 0$, and hence $\tilde{\xi}(-z) > 0$), we have

$$\begin{aligned}\xi(z) &\times (\text{Number of } \mathbf{z}^N \in \mathcal{Z}^N \text{ such that } \overline{\mathbf{z}^N} = \xi)\\ &= \tilde{\xi}(-z) \times (\text{Number of } \mathbf{z}^N \in \mathcal{Z}^N \text{ such that } \overline{\mathbf{z}^N} = \tilde{\xi}).\end{aligned} \tag{2.21}$$

Let $\mathbf{z}_\xi^N \in \mathcal{Z}^N$ (resp. $\tilde{\mathbf{z}}_{\tilde{\xi}}^N \in \mathcal{Z}^N$) be such that $\overline{\mathbf{z}_\xi^N} = \xi$ (resp. $\overline{\tilde{\mathbf{z}}_{\tilde{\xi}}^N} = \tilde{\xi}$); i.e., the configuration $\tilde{\mathbf{z}}_{\tilde{\xi}}^N \in \mathcal{Z}^N$ is obtained from $\mathbf{z}_\xi^N \in \mathcal{Z}^N$ by a particle transition $z \to (-z)$. Noting that $\pi(\mathbf{z}^N)$ depends only on $\overline{\mathbf{z}^N}$, we have

$$\begin{aligned}\wp^N(\xi)N\xi(z)\lambda_{z,(-z)}(\xi) &= \pi_N(\mathbf{z}_\xi^N)(\text{Number of } \mathbf{z}^N \in \mathcal{Z}^N \text{ such that } \overline{\mathbf{z}^N} = \xi)N\xi(z)\lambda_{z,(-z)}(\xi)\\ &= \pi_N(\mathbf{z}_\xi^N)(\text{Number of } \mathbf{z}^N \in \mathcal{Z}^N \text{ such that } \overline{\mathbf{z}^N} = \tilde{\xi})N\tilde{\xi}(-z)\lambda_{z,(-z)}(\xi)\\ &= \pi_N(\mathbf{z}_{\tilde{\xi}}^N)(\text{Number of } \mathbf{z}^N \in \mathcal{Z}^N \text{ such that } \overline{\mathbf{z}^N} = \tilde{\xi})N\tilde{\xi}(-z)\lambda_{(-z),z}(\tilde{\xi})\\ &= \wp^N(\tilde{\xi})N\tilde{\xi}(-z)\lambda_{(-z),z}(\tilde{\xi}),\end{aligned}$$

where we have used (2.21) in the second equality and (2.20) in the third equality. It follows that μ^N is reversible.

Remark 2.6. Another situation where μ^N is reversible with respect to $\wp^N$ is in the non-interacting case (i.e. when, for each $(z, z') \in \mathcal{E}$, $\lambda_{z,z'}(\cdot)$ is a constant function, which we denote by $\lambda_{z,z'}$) where the Markov process on $\mathcal{Z}$ with generator

$$f \mapsto \sum_{z':(z,z')\in\mathcal{E}} (f(z') - f(z))\lambda_{z,z'}, z \in \mathcal{Z}$$

is reversible with respect to its invariant measure (i.e. when the Markov process corresponding to a single particle's evolution on $\mathcal{Z}$ is reversible with respect to its invariant measures). This results in a reversible empirical measure process μ^N. However, the authors are not aware of a general condition (in terms of the transition rates $\lambda_{z,z'}(\cdot), (z, z') \in \mathcal{E}$) that characterises reversibility of μ^N.

2.5 Convergence to a global minimum via controlled addition of particles

In this section, our goal is to increase the number of particles N over time so as to obtain, with high probability, convergence of the empirical measure process to a global minimum of the rate function s that governs the LDP for the sequence of invariant measure $\{\wp^N, N \geq 1\}$.

Fix $c > 0$. Let $N_0 = \min\{n \in \mathbb{N} : \exp\{nc\} - 2 \geq 0\}$, $t_{N_0} = 0$, and for each $N > N_0$, let $t_N = \exp\{Nc\} - 2$. For each $N \geq N_0$ define the generator L_t^N acting on bounded measurable functions on $\mathcal{M}_1(\mathcal{Z})$ by

$$L_t^N f(\xi) := \sum_{(z,z')\in E} N_t \xi(z)\lambda_{z,z'}(\xi) \left[f\left(\xi + \frac{e_{z'}}{N_t} - \frac{e_z}{N_t}\right) - f(\xi)\right], \, t \in [t_N, t_{N+1}).$$

where $N_t = N$ for $t \in [t_N, t_{N+1})$. Let $z_0 \in \mathcal{Z}$ be a fixed state and let $\nu \in \mathcal{M}_1^{N_0}(\mathcal{Z})$. We say that a probability measure $P_{0,\nu}$ on $D([0,\infty), \mathcal{M}_1(\mathcal{Z}))$ is a solution to the martingale problem for $\{L^N, N \geq N_0\}$ with initial condition ν if $P_{0,\nu}(\bar{\mu} : \bar{\mu}(0) = \nu) = 1$; for each $N \geq N_0$, the restriction of $P_{0,\nu}$ on $D([t_N, t_{N+1}), \mathcal{M}_1^N(\mathcal{Z}))$ is a solution to the $D([t_N, t_{N+1}), \mathcal{M}_1^N(\mathcal{Z}))$-valued martingale problem for L^N; and

$$P_{0,\nu}\left(\bar{\mu} : \bar{\mu}(t_{N+1}) = \frac{N}{1+N}\bar{\mu}(t_{N+1}^-) + \frac{1}{N+1}\delta_{z_0}\right) = 1.$$

Again, by the boundedness assumption on transition rates (A2), for each $\nu \in \mathcal{M}_1^{N_0}(\mathcal{Z})$, there exists a unique probability measure $P_{0,\nu}$ that solves the martingale problem for $\{L^N, N \geq N_0\}$ with initial condition ν. Let $\bar{\mu}$ be the process on $D([0,\infty), \mathcal{M}_1(\mathcal{Z}))$ whose law is $P_{0,\nu}$. To describe the process in words, we start with N_0 particles and follow the mean-field interaction described in Section 2.1.1, except that at each time instant $t_N, N > N_0$, we add a new particle whose state is set to z_0. Similarly, for $t > 0$ and $\nu \in \mathcal{M}_1^{\lfloor \log(t+2) \rfloor}$, let $P_{t,\nu}$ denote the law of the process $(\bar{\mu}(t'), t' \geq t)$ with $\bar{\mu}(t) = \nu$.

We anticipate that if c is small then N_t is so large that the fluid limit kicks in too quickly over time and the process $\bar{\mu}$ converges (over time) to a local minimum of s with positive probability depending on the initial condition $\bar{\mu}(0)$. When c is sufficiently large, we anticipate that there is enough time for exploration and therefore we will converge to a global minimum of s. Recall that the set of global minimisers of s is denoted by $\tilde{L}_0$. Our interest in this section is in finding a constant c^* such that for all $c > c^*$ and $\nu \in \mathcal{M}_1^{N_0}(\mathcal{Z})$, we have,

$$P_{0,\nu}(\bar{\mu}(t) \text{ lies in a neighbourhood of } \tilde{L}_0) \to 1 \tag{2.22}$$

as $t \to \infty$.

We use the results in the previous sections to identify the constant c^*. Since $N_t \to \infty$ as $t \to \infty$, for a fixed $T > 0$ and large enough t, the large deviation properties of the process $(\bar{\mu}(s), s \in [t, t+T])$ from the limiting dynamics (2.1) starting at an arbitrary $\bar{\mu}(t)$ can be obtained similar to the LDP for the process μ^N studied in Theorem 2.5 and Corollary 2.1. Therefore, the results in the previous sections on the large time behaviour for the process $(\mu^N(t), t \geq 0)$ are also valid for $(\bar{\mu}(t), t \geq 0)$ when time t is large enough; we make these precise now.

Lemma 2.12 (see Lemma 2.8)**.** *Let π_1^k and π_2^k be k-cycles and suppose that $\pi_1^k \to \pi_2^k$ and $\tilde{V}(\pi_1^k)/c < 1$. Then, given $\varepsilon > 0$, there exist $\delta > 0$ and $\rho > 0$ such that for all $\rho_1 < \rho$, there is $t^* > 0$ such that*

$$P_{t,\nu}(\bar{\tau}_{\pi_1^k} \leq t + t^{(\tilde{V}(\pi_1^k)-\delta)/c}, \bar{\mu}(\bar{\tau}_{\pi_1^k}) \in \gamma_{\pi_2^k}) \geq t^{-\varepsilon/c}$$

holds uniformly for all $\nu \in [\pi_1^k]_{\rho_1} \cap \mathcal{M}_1^{N_t}(\mathcal{Z})$ and $t \geq t^*$.

Remark 2.7. The condition $\tilde{V}(\pi_1^k)/c < 1$ in the above lemma ensures that during the time duration $[t, t + t^{\tilde{V}(\pi_1^k)/c}]$, for large enough t, the number of particles does not change so that Lemma 2.8 for the process μ^N is applicable for the process $\bar{\mu}$.

Lemma 2.13 (see Lemma 2.9)**.** *Let π^k be a k-cycle and suppose that $\tilde{V}(\pi^k)/c < 1$. Then, given $\delta > 0$ such that $(\tilde{V}(\pi^k) + \delta)/c < 1$, there exist $\varepsilon > 0$ and $\rho > 0$ such that for all $\rho_1 < \rho$, there* *is* $t^* > 0$ *such that*

$$P_{t,\nu}(\bar{\tau}_{\pi^k} < t + t^{(\tilde{V}(\pi^k)-\delta)/c}) \leq t^{-\varepsilon/c}, \text{ and}$$
$$P_{t,\nu}(\bar{\tau}_{\pi^k} > t + t^{(\tilde{V}(\pi^k)+\delta)/c}) \leq t^{-\varepsilon/c}$$

hold uniformly for all $\nu \in [\pi^k]_{\rho_1} \cap \mathcal{M}_1^{N_t}(\mathcal{Z})$ *and* $t \geq t^*$.

Lemma 2.14 (see Lemma 2.10)**.** *Let* π^k *be a* k*-cycle and suppose that* $\hat{V}(\pi^k)/c < 1$*. Given* $\varepsilon > 0$*, there exist* $\delta \in (0, c - \hat{V}(\pi^k))$ *and* $\rho > 0$ *such that for all* $\rho_1 \leq \rho$*, there is* $t^* > 0$ *such that*

$$P_{t,\nu}(\bar{\tau}_{\pi^k} \leq t + t^{(\hat{V}(\pi^k)+\delta)/c}) \leq t^{-(\tilde{V}(\pi^k)-\hat{V}(\pi^k)-\varepsilon)/c}$$

holds uniformly for all $\nu \in [\pi^k]_{\rho_1} \cap \mathcal{M}_1^{N_t}(\mathcal{Z})$ *and* $t \geq t^*$.

Recall the definition of the sets L and C from Section 2.3.

Lemma 2.15 (see Lemma 2.3)**.** *Given* $\rho_0 > 0$ *and* $\rho_1 < \rho_0$ *and their associated sets* L *and* C*, given* $v > 0$*, there exist* $T^* > 0$ *and* $t^* > 0$ *such that*

$$P_{t,\nu}(\hat{\tau}_L \geq t + T^*) \leq t^{-v/c}$$

holds uniformly for all $\nu \in C \cap \mathcal{M}_1^{N_t}(\mathcal{Z})$ *and* $t \geq t^*$.

To answer the question on the convergence of $\bar{\mu}$ to a global minimum of s, we define the following quantities, analogous to what is done in Hwang and Sheu [44]. Let m be such that L_{m+1} is a singleton (denote it by $\{\pi^{m+1}\}$). Define

$$A_m := \{\pi^m \in L_m : \tilde{V}(\pi^m) = \hat{V}(\pi^{m+1})\}.$$

Inductively define, for each $\pi^{k+1} \in L_{k+1}$,

$$A_k(\pi^{k+1}) := \{\pi^k \in \pi^{k+1} : \tilde{V}(\pi^k) = \hat{V}(\pi^{k+1})\},$$

and for each $k \geq 1$, define

$$A_k := \bigcup_{\pi^{k+1} \in A_{k+1}} A_k(\pi^{k+1}).$$

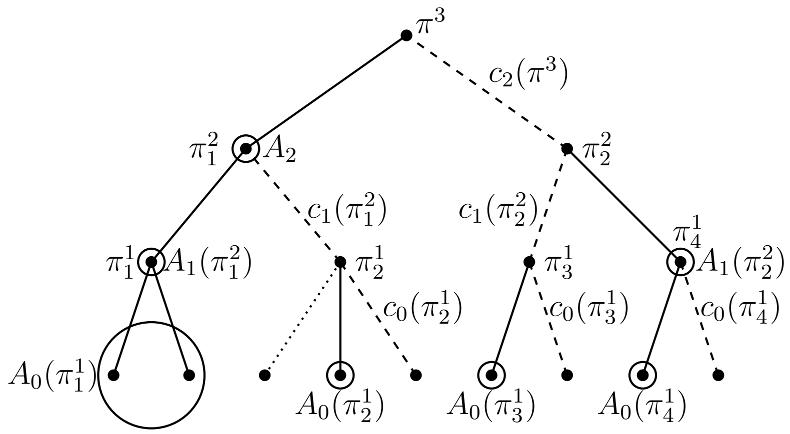

Figure 2.1: An example of the hierarchy of cycles with $|L| = 9$ and $m = 2$ that depicts the above definitions. There are four 1-cycles and two 2-cycles. The nodes in the bottom row represent the elements of L, and the nodes above it represent the hierarchy of cycles. Dashed and dotted lines indicate the $(k-1)$-cycles belonging to a k-cycle. Thick lines indicate the $(k-1)$-cycles that attain $\max_{\pi^{k-1}\in\pi^k} \tilde{V}(\pi^{k-1})$ for a k-cycle π^k. Circles indicate the sets $A_k(\pi^{k+1})$. Dashed lines indicate the cycle $\pi^{k-1} \notin A_{k-1}(\pi^k)$ that attains the maximum in the second line in the definition of $c_{k-1}(\pi^k)$; $c_0(\pi_1^1) = 0$ (which is not indicated in the figure), all other $c_{k-1}(\pi^k)$ are positive.

Also, for each $\pi^k \in L_k$, define

$$c_{k-1}(\pi^k) := \begin{cases} 0, \text{ if } \{\pi^{k-1} \in \pi^k : \pi^{k-1} \notin A_{k-1}(\pi^k)\} = \emptyset, \\ \max\{\tilde{V}(\pi^{k-1}) : \pi^{k-1} \notin A_{k-1}(\pi^k), \pi^{k-1} \in \pi^k\}, \text{ otherwise,} \end{cases}$$

and for each $k \geq 1$, define

$$c_{k-1} := \max\{c_{k-1}(\pi^k), : \pi^k \in A_k\}.$$

Finally, define

$$c^* := \max\{c_k, 0 \leq k \leq m\}.$$

See Figure 2.1 that depicts these definitions. Similar to [44, Lemma A.11, Appendix], we can show that $A_0 = \tilde{L}_0$, the set of minimisers of the rate function s that governs the LDP for the invariant measure $\{\wp^N, N \geq 1\}$. We now prove Theorem 2.4 on convergence of $\bar{\mu}$ to the set of global minimisers.

Proof of Theorem 2.4. Since $\tilde{L}_0 \neq L$, we have that $c^* > 0$. It suffices to show that, for any

$\delta > 0$ with $(c^* + \delta)/c < 1$, there exist $\varepsilon > 0$, $\rho_1 > 0$ and $t^* > 0$ such that

$$P_{t,\nu}(\bar{\mu}(t + t^{(c^*+\delta)/c}) \in [\tilde{L}_0]_{\rho_1}) \geq 1 - t^{-\varepsilon/c}$$

for all $t > t^*$ and $\nu \in \mathcal{M}_1^{N_t}(\mathcal{Z})$. Define the stopping time

$$\theta := \inf\{s > t : \bar{\mu}(s) \in [L]_{\rho_1}\}.$$

By Lemma 2.15, for any $M > 0$, there exists $T^* > 0$ such that for all $\nu \in \mathcal{M}_1^{N_0}(\mathcal{Z})$ and large enough t, we have

$$P_{t,\nu}(\theta > t + T^*) \leq t^{-M/c}.$$

By the strong Markov property, we have

$$\begin{aligned} &P_{t,\nu}(\bar{\mu}(t + t^{(c^*+\delta)/c}) \in [\tilde{L}_0]_{\rho_1}) \\ &\quad \geq E_{t,\nu}(\mathbf{1}_{\{\theta \leq t+T^*\}} \cdot E_{\theta,\bar{\mu}(\theta)}(\mathbf{1}_{\{\bar{\mu}(t+t^{(c^*+\delta)/c}) \in [\tilde{L}_0]_{\rho_1}\}})) \\ &\quad \geq \inf_{\substack{t \leq t_1 \leq t+T^* \\ \nu_1 \in [L]_{\rho_1}}} P_{t_1,\nu_1}(\bar{\mu}(t + t^{(c^*+\delta)/c}) \in [\tilde{L}_0]_{\rho_1})(1 - t^{-M/c}). \end{aligned} \tag{2.23}$$

To bound the first term above, fix a t_1 such that $t \leq t_1 \leq t + T^*$ and $\nu_1 \in [L]_{\rho_1}$. Define the stopping time $\theta_m := \inf\{t > t_1 : \bar{\mu}(t) \in [A_m]_{\rho_1}\}$. We have

$$\begin{aligned} &P_{t_1,\nu_1}(\bar{\mu}(t + t^{(c^*+\delta)/c}) \in [\tilde{L}_0]_{\rho_1}) \\ &\quad \geq E_{t_1,\nu_1}(\mathbf{1}_{\{\theta_m < t+t^{(c^*+\delta/2)/c}\}} \cdot E_{\theta_m,\bar{\mu}(\theta_m)}(\mathbf{1}_{\{\bar{\mu}(t+t^{(c^*+\delta)/c}) \in [\tilde{L}_0]_{\rho_1}\}})) \\ &\quad \geq \inf_{t \leq t_2 \leq t+t^{(c^*+\delta/2)/c}, \nu_2 \in [A_m]_{\rho_1}} P_{t_2,\nu_2}(\bar{\mu}(t + t^{(c^*+\delta)/c}) \in [\tilde{L}_0]_{\rho_1}) \\ &\qquad \times P_{t_1,\nu_1}(\theta_m \leq t + t^{(c^*+\delta/2)/c}). \end{aligned} \tag{2.24}$$

We first bound the second term $P_{t_1,\nu_1}(\theta_m \leq t + t^{(c^*+\delta/2)/c})$. Note that, by Lemma 2.12, for any $M_1 > 0$, there exists $\delta_1 > 0$ such that

$$P_{t_1,\nu_1}(\theta_m > t_1 + t_1^{(c_m - \delta_1)/c}) \leq 1 - t_1^{-M_1/c}$$

for sufficiently large t. Let $T_1 = t_1 + t_1^{(c_m-\delta_1)/c}$, and define the stopping time $\hat{\theta} := \inf\{t > T_1 : \bar{\mu}(t) \in [L]_{\rho_1}\}$. Again, by Lemma 2.15, there exists a large enough T^* such that $P_{T_1,\nu}(\hat{\theta} >$

$T_1 + T^*) \leq T_1^{-M/c}$ for all $\nu \in M_1^{N_{T_1}}(\mathcal{Z})$. Therefore, using the strong Markov property, we have

$$\begin{aligned}
&P_{t_1,\nu_1}(\theta_m > t + t^{(c^*+\delta/2)/c}) \\
&\quad \leq E_{t_1,\nu_1}(\mathbf{1}_{\{\theta_m \geq \hat{\theta}, \hat{\theta} < T_1 + T^*\}} \cdot E_{\hat{\theta},\bar{\mu}(\hat{\theta})}(\mathbf{1}_{\{\theta_m > t + t^{(c^*+\delta/2)/c}\}})) \\
&\qquad + P_{t_1,\nu_1}(\hat{\theta} > T_1 + T^*) \\
&\quad \leq P_{t_1,\nu_1}(\theta_m > T_1) \sup_{\substack{T_1 \leq t \leq T_1 + T^* \\ \nu \in [L]_{\rho_1}}} P_{t,\nu}(\theta_m > t + t^{(c^*+\delta/2)/c}) + t_1^{-M/c} \\
&\quad \leq (1 - t_1^{-M_1/c}) \sup_{\substack{T_1 \leq t \leq T_1 + T^* \\ \nu \in [L]_{\rho_1}}} P_{t,\nu}(\theta_m > t + t^{(c^*+\delta/2)/c}) + t_1^{-M/c}. \qquad (2.25)
\end{aligned}$$

We now focus on $P_{t,\nu}(\theta_m > t + t^{(c^*+\delta/2)/c})$ for a fixed $t \in [T_1, T_1 + T^*]$ and $\nu \in [L]_{\rho_1}$, and repeat the above steps; this will introduce a multiplication factor of $(1 - T_1^{-M_1/c})$ along with

$$\sup_{\substack{T_2 \leq t \leq T_2 + T^* \\ \nu \in [L]_{\rho_1}}} P_{t,\nu}(\theta_m > t + t^{(c^*+\delta/2)/c}),$$

where $T_2 = T_1 + T_1^{(c_m - \delta_1)/c}$, in the first term in (2.25), and an addition of $t_1^{-M/c}$ in the second term. Therefore, repeating the above steps $r \sim t_1^{\delta/2c}$ times, we get

$$P_{t_1,\nu_1}(\theta_m > t + t^{(c^*+\delta/2)/c}) \leq \prod_{n=0}^{r} (1 - T_n^{*-M_1/c}) + r t_1^{-M/c},$$

where $T_0^* = t_1$, and

$$T_{n+1}^* = T_n^* + T_n^{*(c_m - \delta_1)/c} + T^*.$$

Note that,

$$\begin{aligned}
\prod_{n=0}^{r} (1 - T_n^{*-M_1/c}) &\leq \exp\left\{ -\sum_{n=0}^{r} T_n^{*-M_1/c} \right\} \\
&= \exp\left\{ -\sum_{n=0}^{r} T_n^{*-M_1/c - (c_m - \delta_1)/c} (T_{n+1}^* - T_n^*) \right\} \\
&\leq \exp\left\{ -\int_{T_0^*}^{T_r^*} u^{-(M_1/c) - (c_m - \delta_1)/c} du \right\} \\
&= \exp\left\{ -\left(T_r^{*1 - (c_m + M_1 - \delta_1)/c} - t_1^{1 - (c_m + M_1 - \delta_1)/c} \right) \right\}. \qquad (2.26)
\end{aligned}$$

Since $T_n \geq t_1$ for all $n \geq 1$, we see that $T_r^* \geq t_1 + rt_1^{(c_m-\delta_1)/c} \sim t_1 + t_1^{(c_m-\delta_1+\delta/2)/c}$. Therefore,

$$\begin{aligned}
&-\left(T_r^{*1-(c_m+M_1-\delta_1)/c} - t_1^{1-(c_m+M_1-\delta_1)/c}\right)\\
&\leq -\left((t_1 + t_1^{(c_m-\delta_1+\delta/2)/c})^{1-(c_m+M_1-\delta_1)/c} - t_1^{1-(c_m+M_1-\delta_1)/c}\right)\\
&\leq -\left(t_1^{1-(c_m+M_1-\delta_1)/c}\left(1 + t_1^{(c_m-\delta_1+\delta/2)/c-1}\right)^{1-(c_m+M_1-\delta_1)/c} - 1\right)\\
&\leq -c'\left(t_1^{1-(c_m+M_1-\delta_1)/c} t_1^{(c_m-\delta_1+\delta/2)/c-1}\right)\\
&= -c' t_1^{(\delta/2-M_1)/c},
\end{aligned}$$

for some constant $c' > 0$ and large enough t_1. Hence, (2.26) becomes

$$\prod_{n=0}^{r}(1 - T_n^{*^{-M_1/c}}) \leq \exp\{-c' t_1^{(\delta/2-M_1)/c}\}.$$

We choose $M_1 = \delta/4$; the above and (2.25) then implies

$$P_{t_1,\nu_1}(\theta_m > t + t^{(c^*+\delta/2)/c}) \leq \exp\{-c' t_1^{\delta/4c}\} + t_1^{-(M-\delta/2)/c},$$

and this implies that, for any $M' > 0$,

$$P_{t_1,\nu_1}(\theta_m > t + t^{(c^*+\delta/2)/c}) \leq t^{-M'/c} \tag{2.27}$$

for sufficiently large t, $t \leq t_1 \leq t + T^*$ and for all $\nu \in [L]_{\rho_1}$.

We now bound the first term in (2.24), $P_{t_2,\nu_2}(\bar{\mu}(t + t^{(c^*+\delta)/c}) \in [\tilde{L}_0]_{\rho_1})$ where $t \leq t_2 \leq t + t^{(c^*+\delta/2)/c}$ and $\nu_2 \in [A_m]_{\rho_1}$. Let $\pi_0^m \in A_m$ be the m-cycle such that $\nu_2 \in [\pi_0^m]_{\rho_1}$. Define the following quantities:

$$\begin{aligned}
\tilde{t}_0 &:= t + t^{(c^*+\delta)/c} - t^{(c_{m-1}(\pi_0^m)+\delta)/c}, \text{ and}\\
\tilde{t}_1 &:= t + t^{(c^*+\delta)/c} - t^{(c_{m-1}(\pi_0^m)+\delta/2)/c}.
\end{aligned}$$

Define the stopping time $\theta := \inf\{t > \tilde{t}_0 : \bar{\mu}(t) \in [\pi_0^m]_{\rho_1}\}$, if $c^* > c_{m-1}(\pi_0^m)$ and $\theta = t_2$ otherwise. By the strong Markov property,

$$\begin{aligned}
&P_{t_2,\nu_2}(\bar{\mu}(t + t^{(c^*+\delta)/c}) \in [\tilde{L}_0]_{\rho_1})\\
&\quad\geq E_{t_2,\nu_2}(\mathbf{1}_{\{\theta \leq \tilde{t}_1\}} \cdot E_{\theta,\bar{\mu}(\theta)}(\mathbf{1}_{\{\bar{\mu}(t+t^{(c^*+\delta)/c}) \in [\tilde{L}_0]_{\rho_1}\}}))
\end{aligned}$$

$$\geq P_{t_2,\nu_2}(\theta \leq \tilde{t}_1) \inf_{\tilde{t}_0 \leq t_3 \leq \tilde{t}_1, \nu_3 \in [\pi_0^m]_{\rho_1}} P_{t_3,\nu_3}(\bar{\mu}(t + t^{(c^*+\delta)/c}) \in [\tilde{L}_0]_{\rho_1}). \tag{2.28}$$

We first estimate $P_{t_2,\nu_2}(\theta \leq \tilde{t}_1)$ when $c^* > c_{m-1}(\pi_0^m)$ (if this is not the case, then by definition of θ, we have $P_{t_2,\nu_2}(\theta \leq \tilde{t}_1) = 1$) . Note that

$$\begin{aligned} P_{t_2,\nu_2}(\theta > \tilde{t}_1) &= P_{t_2,\nu_2}(\bar{\mu}(t) \notin [\pi_0^m]_{\rho_1} \text{ for all } \tilde{t}_0 \leq t \leq \tilde{t}_1) \\ &\leq P_{t_2,\nu_2}(\bar{\mu}(t) \notin [L]_{\rho_1} \text{ for all } \tilde{t}_0 \leq t \leq \tilde{t}_1) + P_{t_2,\nu_2}(\bar{\tau}_{\pi_0^m} \leq \tilde{t}_1). \end{aligned}$$

Lemma 2.13 implies that

$$P_{t_2,\nu_2}(\bar{\tau}_{\pi_0^m} \leq \tilde{t}_1) \leq t^{-\delta/c}$$

for large t and small enough $\rho_1 > 0$. Also, with this ρ_1, by using Lemma 2.15, we see that

$$P_{t_2,\nu_2}(\bar{\mu}(t) \notin [L]_{\rho_1} \text{ for all } \tilde{t}_0 \leq t \leq \tilde{t}_1) \leq t^{-M_1/c}$$

for large t, where M_1 can be chosen as large as we want. This shows that there exists $\varepsilon_1 > 0$ such that

$$P_{t_2,\nu_2}(\theta \leq \tilde{t}_1) \geq 1 - 2t^{-\varepsilon_1/c}$$

uniformly for all $\nu_2 \in [\pi_0^m]_{\rho_1}$ and large enough t. Hence, from (2.27), (2.28) and (2.24), we get

$$\begin{aligned} &P_{t_1,\nu_1}(\bar{\mu}(t + t^{(c^*+\delta)/c}) \in [\tilde{L}_0]_{\rho_1}) \\ &\quad\geq (1 - t^{-M'/c})(1 - 2t^{-\varepsilon_1/c}) \times \inf_{\substack{t_2 \geq \tilde{t}_0, \\ \nu_2 \in [\pi_0^m]_{\rho_1} \\ \pi_0^m \in A_m \\ \tilde{\delta} \in [\delta/4, \delta]}} P_{t_2,\nu_2}(\bar{\mu}(t_2 + t_2^{(c_{m-1}(\pi_0^m)+\tilde{\delta})/c}) \in [\tilde{L}_0]_{\rho_1}) \end{aligned}$$

and therefore, for some $\varepsilon > 0$, we have

$$\begin{aligned} &\inf_{\substack{t \leq t_1 \leq t+T^*, \\ \nu_1 \in [L]_{\rho_1}}} P_{t_1,\nu_1}(\bar{\mu}(t + t^{(c^*+\delta)/c}) \in [\tilde{L}_0]_{\rho_1}) \\ &\qquad\geq (1 - t^{-\varepsilon/c}) \times \inf_{\substack{t_2 \geq \tilde{t}_0 \\ \nu_2 \in [\pi_0^m]_{\rho_1} \\ \pi_0^m \in A_m \\ \tilde{\delta} \in [\delta/4, \delta]}} P_{t_2,\nu_2}(\bar{\mu}(t_2 + t_2^{(c_{m-1}(\pi_0^m)+\tilde{\delta})/c}) \in [\tilde{L}_0]_{\rho_1}). \end{aligned}$$

We now focus on the second term. This probability inside the infimum can be lower bounded using similar steps above starting with (2.28); instead of the random variable θ, we consider the hitting time of a suitable $(m-1)$-cycle. Continuing this procedure m times, we eventually reach A_0. Therefore, we can show

$$\inf_{\substack{t \leq t_1 \leq t+T^* \\ \nu_1 \in [\tilde{L}_0]_{\rho_1}}} P_{t_1,\nu_1}(\bar{\mu}(t + t^{(c^*+\delta)/c}) \in [\tilde{L}_0]_{\rho_1}) \geq (1 - t^{-\varepsilon/c})^{m+1},$$

and the result now follows from (2.23). □

We now show that the conclusion of Theorem 2.4 fails if we choose $c < c^*$. Since $\tilde{L}_0 \neq L$, we have $c^* > 0$. Given $c < c^*$, let $\pi^k \in L_k$ be such that $\hat{V}(\pi^k) \leq c < \tilde{V}(\pi^k)$; this is possible from the definition of c^*. Note that $\tilde{L}_0 \cap \pi^k = \emptyset$. The below result shows that the exit time from a neighbourhood of π^k is infinite with positive probability, and this in particular implies that (2.22) fails.

Proposition 2.1. *Let π^k be a k-cycle such that $\hat{V}(\pi^k) \leq c < \tilde{V}(\pi^k)$. There exist $\varepsilon \in (0, \tilde{V}(\pi^k) - c)$, $c' > 0$, $\rho_1 > 0$, and $t^* > 0$ such that for all $\nu \in [\pi^k]_{\rho_1} \cap \mathcal{M}_1^{N_t}(\mathcal{Z})$ and $t \geq t^*$, we have*

$$P_{t,\nu}(\bar{\tau}_{\pi^k} < \infty) \leq c' t^{1-(\tilde{V}(\pi^k)-\varepsilon)/c}.$$

Proof. We proceed via the steps in Hwang and Sheu [44]. Let $T_0 = t$, and define, for all $n \geq 1$,

$$T_{n+1} \coloneqq T_n + T_n^{\hat{V}(\pi^k)/c}, \text{ and}$$
$$T_{n+1}^* \coloneqq T_n + \frac{1}{2} T_n^{\hat{V}(\pi^k)/c}.$$

(In the above definitions, we assume that $\hat{V}(\pi^k) > 0$; if this is not the case, then we replace $T_n^{\hat{V}(\pi^k)/c}$ in the above definitions by a sufficiently large constant, and the following arguments will go through.) We have, for any $r \geq 1$,

$$P_{t,\nu}(\bar{\tau}_{\pi^k} < T_r) = P_{t,\nu}(\bar{\tau}_{\pi^k} < T_{r-1}) + P_{t,\nu}(T_{r-1} \leq \bar{\tau}_{\pi^k} < T_r). \tag{2.29}$$

To bound the second term, define the stopping time $\theta \coloneqq \inf\{t > T_{r-1}^* : \bar{\mu}(t) \in [L]_{\rho_1}\}$ where ρ_1 is to be chosen later. Then,

$$\begin{aligned} P_{t,\nu}(T_{r-1} \leq \bar{\tau}_{\pi^k} < T_r) &= P_{t,\nu}(T_{r-1} \leq \bar{\tau}_{\pi^k} < T_r, \theta \leq T_{r-1}^* + T^*) \\ &\quad + P_{t,\nu}(T_{r-1} \leq \bar{\tau}_{\pi^k} < T_r, \theta > T_{r-1}^* + T^*), \end{aligned} \tag{2.30}$$

where T^* is such that the second term above is upper bounded by $T_{r-1}^{*-M/c}$ for some $M > 0$ to be chosen later (this is possible by Lemma 2.15). To bound the first term, note that

$$\begin{aligned}
P_{t,\nu}(T_{r-1} \leq &\bar{\tau}_{\pi^k} < T_r, \theta \leq T_{r-1}^* + T^*) \\
&\leq P_{t,\nu}(\theta \leq \bar{\tau}_{\pi^k} < T_r, \theta \leq T_{r-1}^* + T^*) \\
&\leq E_{t,\nu}(\mathbf{1}_{\{\bar{\mu}(\theta)\in[\pi^k]_{\rho_1}, \theta \leq T_{r-1}^*+T^*\}} \cdot E_{\theta,\bar{\mu}(\theta)}(\mathbf{1}_{\{\bar{\tau}_{\pi^k}<T_r\}})) \\
&\leq T_{r-1}^{*-(\tilde{V}(\pi^k)-\hat{V}(\pi^k)-\varepsilon)/c}
\end{aligned}$$

holds for sufficiently large t and small enough ρ_1. Here, the second inequality follows by the strong Markov property and the third from Lemma 2.14. Choose M sufficiently large, so that (2.29), (2.30) and the above implies

$$P_{t,\nu}(\bar{\tau}_{\pi^k} < T_r) \leq P_{t,\nu}(\bar{\tau}_{\pi^k} < T_{r-1}) + 2T_{r-1}^{*-(\tilde{V}(\pi^k)-\hat{V}(\pi^k)-\varepsilon)/c}.$$

Therefore, we have

$$\begin{aligned}
P_{t,\nu}(\bar{\tau}_{\pi^k} < T_r) &\leq 2\sum_{n=0}^{r} T_n^{*-(\tilde{V}(\pi^k)-\hat{V}(\pi^k)-\varepsilon)/c} \\
&\leq c_1' \sum_{n=0}^{r} T_n^{-(\tilde{V}(\pi^k)-\hat{V}(\pi^k)-\varepsilon)/c} \\
&= c_1' \sum_{n=0}^{r} T_n^{-(\tilde{V}(\pi^k)-\varepsilon)/c}(T_{n+1} - T_n) \\
&\leq c_1' \int_t^{T_r} u^{-(\tilde{V}(\pi^k)-\varepsilon)/c} du,
\end{aligned}$$

where c_1' is a positive constant. Choose ε such that $\tilde{V}(\pi^k) - \varepsilon > c$ so that the above implies

$$\begin{aligned}
P_{t,\nu}(\bar{\tau}_{\pi^k} < T_r) &\leq c_1' \int_t^{\infty} u^{-(\tilde{V}(\pi^k)-\varepsilon)/c} du \\
&\leq c' t^{1-(\tilde{V}(\pi^k)-\varepsilon)/c},
\end{aligned}$$

where c' is a positive constant. Let $r \to \infty$, and the result follows since $T_r \to \infty$. □

Example 2.3. We now provide an example where we can choose the transition rates of the particles so as to minimise a given "nice" function U on $\mathcal{M}_1(\mathcal{Z})$. Let $(\mathcal{Z}, \mathcal{E}_{\mathcal{Z}})$ denote the complete graph on $\mathcal{Z}$. Suppose that for every $\xi \in \mathcal{M}_1(\mathcal{Z})$ and $(z, z') \in \mathcal{E}_{\mathcal{Z}}$ with $\xi(z) > 0$, the limit

$$\nabla_{z,z'}U(\xi) = \lim_{N\to\infty} \frac{U\left(\xi + \frac{\delta_{z'}-\delta_z}{N}\right) - U(\xi)}{1/N}$$

exists, and is bounded and continuous. Further, assume that the above convergence is uniform over ξ. Consider the transition rates

$$\lambda^{(N)}_{z,z'}(\xi) = \frac{\exp\left\{ - N\left(U\left(\xi + \frac{\delta_{z'}-\delta_z}{N}\right) - U(\xi)\right)\right\}}{1+\exp\left\{ - N\left(U\left(\xi + \frac{\delta_{z'}-\delta_z}{N}\right) - U(\xi)\right)\right\}}, \xi \in \mathcal{M}_1^N(\mathcal{Z}), \xi(z) > 0.$$

Then, by verifying the detailed balance condition, it is straightforward to show that the probability measure

$$\frac{1}{c_N}\exp\{-NU(\overline{\mathbf{z}^N})\}, \mathbf{z}^N \in \mathcal{Z}^N,$$

is invariant for the N-particle evolution, where $c_N = \sum_{\mathbf{z}^N\in\mathcal{Z}^N}\exp\{-NU(\overline{\mathbf{z}^N})\}$. Let $H : \mathcal{M}_1(\mathcal{Z}) \to [0,\infty)$ be the Shannon entropy defined by $H(\xi) = -\sum_{z\in\mathcal{Z}}\xi(z)\log\xi(z)$, with the convention that $0\log 0 = 0$. Since the number of $\mathbf{z}^N \in \mathcal{Z}^N$ such that $\overline{\mathbf{z}^N} = \xi$ is between $(N+1)^{-|\mathcal{Z}|}\exp\{NH(\xi)\}$ and $\exp\{NH(\xi)\}$ [29, Lemma 2.1.8], $\wp^N$ satisfies

$$\frac{(N+1)^{-|\mathcal{Z}|}}{c_N}\exp\{-N(U(\xi)-H(\xi))\} \le \wp^N(\xi) \le \frac{1}{c_N}\exp\{-N(U(\xi)-H(\xi))\}.$$

Therefore, $\{\wp^N\}$ satisfies the LDP with rate function $U - H$. Noting that $\lambda^{(N)}_{z,z'}(\xi)$ converges to $\lambda_{z,z'}(\xi) = \frac{\exp\{-\nabla_{z,z'}U(\xi)\}}{1+\exp\{-\nabla_{z,z'}U(\xi)\}}$ as $N\to\infty$ uniformly over ξ, the empirical measure process μ^N satisfies the process-level LDP, see [50]. Therefore, by modifying U to cU, $c>0$, the particle addition algorithm could ensure convergence to a small neighbourhood of a global minimum of $cU - H$. By choosing c large enough, we can ensure convergence to a neighbourhood of a global minimum of U.

Chapter 3

Process-Level Large Deviations of Two Time Scale Mean-Field Models

3.1 The setting and main results

3.1.1 Introduction

Let $\mathcal{X}, \mathcal{Y}$ be finite sets and $(\mathcal{X}, \mathcal{E}_{\mathcal{X}})$ and $(\mathcal{Y}, \mathcal{E}_{\mathcal{Y}})$ be directed graphs on $\mathcal{X}$ and $\mathcal{Y}$ respectively. Let $\mathcal{M}_1(\mathcal{X})$ denote the space of probability measures on $\mathcal{X}$. For each $N \geq 1$, we consider Markov processes with infinitesimal generators acting on functions f on $\mathcal{M}_1^N(\mathcal{X}) \times \mathcal{Y}$ of the form

$$\sum_{(x,x') \in \mathcal{E}_{\mathcal{X}}} N\xi(x)\lambda_{x,x'}(\xi, y) \left[f\left(\xi + \frac{\delta_{x'}}{N} - \frac{\delta_x}{N}, y\right) - f(\xi, y) \right] + N \sum_{y' : (y,y') \in \mathcal{E}_{\mathcal{Y}}} (f(\xi, y') - f(\xi, y))\gamma_{y,y'}(\xi), \ \xi \in \mathcal{M}_1^N(\mathcal{X}) \text{ and } y \in \mathcal{Y};$$

here $\mathcal{M}_1^N(\mathcal{X}) \subset \mathcal{M}_1(\mathcal{X})$ denotes the set of probability measures on $\mathcal{X}$ that can arise as empirical measures of N-particle configurations on $\mathcal{X}^N$, $\lambda_{x,x'}(\cdot, y) : \mathcal{M}_1(\mathcal{X}) \to \mathbb{R}_+$, $(x, x') \in \mathcal{E}_{\mathcal{X}}$ and $y \in \mathcal{Y}$, and $\gamma_{y,y'} : \mathcal{M}_1(\mathcal{X}) \to \mathbb{R}_+$, $(y, y') \in \mathcal{E}_{\mathcal{Y}}$, are given functions. Such processes arise in the context of weakly interacting Markovian mean-field particle systems in a fast varying environment where the empirical measure of the particle system evolves in the slow time scale and the environment process evolves in the fast time scale. An important feature of such processes is that they are "fully coupled", i.e., the evolution of the empirical measure depends on the state of the environment, and the environment itself changes its state depending on the empirical measure of the particle system. This chapter establishes a process-level large deviation principle (LDP) for the joint law of the empirical measure process and the occupation measure of the fast environment for such fully coupled two time scale mean-field models (see Section 3.1.3 for the precise mathematical model and Theorem 3.1 for the statement of the main result).

Our study of the LDP for such a two time scale mean-field model is motivated by the metastability phenomenon in networked systems. Many networked systems that arise in practice can be modelled using a two time scale mean-field model; see Section 1.1.2 for details of a wireless local area network. In such networks, there could be multiple seemingly "stable points of operation", or metastable points. Some of these may be desirable but some others undesirable in terms of some performance metrics. One is often interested in understanding the following metastable phenomena: (i) the mean time spent by the network near an operating point, (ii) the mean time required for transiting from one stable operating point to another, (iii) the mean time for the system to be sufficiently close to stationarity, etc. The process-level large deviations result established in this chapter, along with the results of Chapter 2, helps to answer such questions on the large time behaviour of these systems when the number of particles N is large.

The above two time scale mean-field model is an example of a stochastic process with time scale separation where a certain component of the process evolves in the slow time scale (i.e. $O(1)$-change in a given $O(1)$ time duration) and another component evolves in the fast time scale (i.e. $\Omega(N)$-change in a given $O(1)$ time duration). Such processes that evolve on multiple time scales have been well studied in the past, and it is known that, under mild conditions, they exhibit the "averaging principle": when the time scale separation N becomes large, the slow component tracks the solution to a certain dynamical system whose driving function is "averaged" over the stationary behaviour of the fast component. In his seminal work, Khasminskii [47] first proved the averaging principle for two time scale diffusions. Freidlin and Wentzell [37, Chapter 7, Section 9] studied the averaging phenomenon in a fully coupled system of diffusions where both the drift and the diffusion coefficients of the slow component depend on the fast component and vice-versa. Their proof is based on discretisation arguments. The averaging phenomenon has also been studied in the context of jump processes with applications to performance analysis of various computer communication systems and queueing networks – Castiel et al. [20] studied a carrier sense multiple access algorithm in the context of wireless networks, Bordenave et al. [14] studied performance analysis of wireless local area networks, Hunt and Kurtz [42] studied scaling limits of loss networks, Hunt and Laws [43] studied analysis of trunk reservation policy in the context of loss networks; also see Kelly [46] and the references therein for other works on loss networks in the two time scale framework. While the above works on jump processes study the averaging principle in the large-N limit, this chapter focuses on the process-level large deviations from the large-N limit.

Various authors have studied process-level large deviations of diffusion processes evolving on multiple time scales under various assumptions – see Freidlin and Wentzell [37], Veretennikov [87, 88], Liptser [57], Puhalskii [73] and the references therein. Liptser [57] established the large deviation principle for the joint law of the slow process and the occupation measure of the fast process for one-dimensional diffusions when the fast process does not depend on the slow variable. More recently, Puhalskii [73] extended this for multidimensional diffusions when the slow and fast processes are fully coupled. His approach is based on the method of stochastic exponentials for large deviations [70], where one identifies a suitable exponential martingale associated with the process and characterises the rate function in terms of this exponential martingale. In identifying the rate function, the main ingredient in the proof is to study a certain variational problem and show certain continuity property of its solution.

In this chapter, our proof of the process-level large deviation result is based on the method of stochastic exponentials, see Puhalskii [70, 73], but the main difficulty lies in extending the approach of Puhalskii [73] to our two time scale mean-field model with jumps. In particular, our setting requires us to study certain variational problems in an Orlicz space, instead of the usual L^2 space in the context of diffusions, to characterise the rate function; see Theorem 3.6 and Theorem 3.8. While Puhalskii [73] uses tools from the theory of elliptic partial differential equations for the characterisation of the rate function, we use tools from convex analysis and parametric continuity of optimisation problems. Also, our mean-field setting makes the solutions to these variational problems blow up near the boundary of the state space, and one of the main novelties of our work is the methodology to obtain a characterisation of the rate function in such cases via suitable approximations – see Section 3.6.

Other works in the two time scale regime include Budhiraja et al. [18] who studied the case where the slow process is a diffusion and the fast process is a Markov chain on a finite set; their proof is based on the weak convergence approach to large deviations where one establishes the LDP by studying certain controlled versions of the processes. Kumar and Popovic [53] established the LDP for two time scale jump-diffusions under some general conditions via convergence of nonlinear semigroups, but their approach requires verification of the comparison principle for a certain nonlinear operator. While this is a possible alternative approach for the mean-field problem under consideration, we have used the more probabilistic stochastic exponentials approach.

Let us also mention some works on large deviations of mean-field models that do not involve the fast environment. Dawson and Gärtner [26] established the process-level large deviations of interacting diffusions of mean-field type where each particle evolves as a diffusion process with coefficients that depend on the other particles via the empirical measure of the states of all the particles. Léonard [56, 55] extended this to the case of jump processes. Our work can be viewed as an extension of Léonard [55] to the case of finite-state mean-field interacting particle systems with a fully coupled fast varying environment. In the stationary regime, Borkar and Sundaresan [15] studied the large deviations of the stationary measure of finite-state mean-field interacting particle systems using tools from Freidlin and Wentzell [37, Chapter 6]. Our results in this chapter, along with the results of Chapter 2, can be used to study the large time behaviour and metastability in two time scale mean-field models; see Section 3.1.4.2.

The rest of this chapter is organised as follows. In the rest of this section, we describe our fully coupled two time scale mean-field model and state our main result and its implications. The proof of the main result is carried out in Sections 3.2–3.7. Section 3.2 establishes the exponential tightness of the joint law of the empirical measure process and the occupation measure process of the fast environment. In Section 3.3, we define a certain exponential martingale and show a necessary condition that holds for every subsequential rate function. In Section 3.4, we define our candidate rate function using the above exponential martingale and study its relevant properties. In Section 3.5, we obtain a characterisation of subsequential rate functions for sufficiently regular elements in the space and Section 3.6 extends this to the whole space using certain approximation arguments. Finally we complete the proof of the main result in Section 3.7.

3.1.2 Notation

We summarise the frequently used notation in this chapter; previously used notation from Chapter 2 are also recalled here for completeness. Let $\langle \cdot, \cdot \rangle$ denote the inner product and $\| \cdot \|$ denote the norm on Euclidean spaces. Given a complete separable metric space $\mathcal{S}$, let $B(\mathcal{S})$ denote the space of bounded Borel-measurable functions on $\mathcal{S}$ equipped with the uniform topology. Let $\mathcal{M}(\mathcal{S})$ denote the space of finite measures on $\mathcal{S}$ equipped with the topology of weak convergence. Let $\mathcal{M}_1(\mathcal{S})$ denote the space of probability measures on $\mathcal{S}$ equipped with the Lévy-Prohorov metric (which generates the topology of weak convergence). (If $\mathcal{S}$ is a finite set, then $\mathcal{M}_1(\mathcal{S})$ can be viewed as an $(|\mathcal{S}|-1)$-dimensional subset of the Euclidean space $\mathbb{R}^{|\mathcal{S}|}$; in this case, for $\nu \in \mathcal{M}_1(\mathcal{S})$, we shall denote the density of ν with respect to the counting measure on $\mathcal{S}$ by ν). Given $N \in \mathbb{N}$, $\mathcal{M}_1^N(\mathcal{S}) \subset \mathcal{M}_1(\mathcal{S})$ denotes the set of probability measures that can arise as empirical measures of N points on $\mathcal{S}$. Given $T > 0$, let $D([0,T], \mathcal{S})$ (resp. $D(\mathbb{R}_+, \mathcal{S})$) denote the space of càdlàg functions on $[0,T]$ (resp. $\mathbb{R}_+$) equipped with the Skorohod-J_1 topology (see, for example, Ethier and Kurtz [34, Chapter 3]). Similarly, given a finite set $\mathcal{Y}$, $D_{\uparrow}([0,T],\mathcal{M}(\mathcal{Y})) \subset D([0,T],\mathcal{M}(\mathcal{Y}))$ denotes the space of càdlàg functions θ on $[0,T]$ such that for each $0 \le s \le t \le T$, $\theta_t - \theta_s$ is an element of $\mathcal{M}(\mathcal{Y})$ and $\theta_t(\mathcal{Y}) = t$. This equipped with its subspace topology is a complete and separable metric space, and is closed in $D([0,T],\mathcal{M}(\mathcal{Y}))$. If X is an element of $D([0,T],\mathcal{S})$, $D([0,\infty),\mathcal{S})$ or $D_{\uparrow}([0,T],\mathcal{M}(\mathcal{Y}))$, let X_t and $X(t)$ denote the coordinate projection of X at time t.

Recall the functions τ and τ^* defined in Section 2.2, i.e., $\tau(u) := e^u - u - 1, u \in \mathbb{R}$, and

$$\tau^*(u) := \begin{cases} +\infty & \text{if } u < -1 \\ 1 & \text{if } u = -1 \\ (u+1)\log(u+1) - u & \text{if } u > -1. \end{cases}$$

Given a complete separable metric space $\mathcal{S}$ and a finite measure ϑ on $\mathcal{S}$, let $L^{\tau}(\mathcal{S},\vartheta)$ and $L^{\tau^*}(\mathcal{S},\vartheta)$ denote the Orlicz spaces corresponding to the functions τ and τ^*, respectively (see, for example, Rao and Ren [76, Chapter 3] for an introduction to Orlicz spaces). The Orlicz norms on these spaces are denoted by $\|\cdot\|_{L^{\tau}(\mathcal{S},\vartheta)}$ and $\|\cdot\|_{L^{\tau^*}(\mathcal{S},\vartheta)}$, respectively. Given a directed and connected graph (V,E) and $\Delta = (u,v) \in E$, let $u + \Delta$ denote v. Given a function f on $[0,T] \times \mathcal{S} \times V$, let Df denote the function on $[0,T] \times \mathcal{S} \times V \times E$ defined as follows: if $(t,s,u,\Delta) \in [0,T] \times \mathcal{S} \times V \times E$ is such that the edge Δ is an outgoing edge from the vertex u, then define $f(t,s,u,\Delta) = f(t,s,v) - f(t,s,u)$ where $v = u + \Delta$; otherwise define $f(t,s,u,\Delta) = 0$. Given a subset W of a Euclidean space and $T > 0$, let $C^{1,1}([0,T] \times W \times \mathcal{S})$ (resp. $C^{\infty}([0,T] \times W \times \mathcal{S})$) denote the space of functions on $f(t,u,s)$, $(t,u,s) \in [0,T] \times W \times \mathcal{S}$, that is continuously differentiable (resp. infinitely differentiable) in both t and u.

We finally recall the definition of a large deviation principle from Definition 1.1. Let $(\mathcal{S}, d_0)$ be a metric space. We say that a sequence $\{X^N, N \ge 1\}$ of $\mathcal{S}$-valued random variables defined on a probability space $(\Omega, \mathcal{F}, P)$ satisfies the large deviation principle (LDP) with rate function $I : \mathcal{S} \to [0,+\infty]$ if

- (Compactness of level sets). For any $s \ge 0$, $\Phi(s) := \{x \in \mathcal{S} : I(x) \le s\}$ is a compact subset of $\mathcal{S}$;
- (LDP lower bound). For any $\gamma > 0$, $\delta > 0$, and $x \in \mathcal{S}$, there exists $N_0 \ge 1$ such that

  $$P(d_0(X^N, x) < \delta) \ge \exp\{-N(I(x) + \gamma)\}$$

  for any $N \ge N_0$;

- (LDP upper bound). For any $\gamma > 0$, $\delta > 0$, and $s > 0$, there exists $N_0 \geq 1$ such that

$$P(d_0(X^N, \Phi(s)) \geq \delta) \leq \exp\{-N(s-\gamma)\}$$

  for any $N \geq N_0$.

We say that $I : \mathcal{S} \to [0, +\infty]$ is a subsequential rate function for the family $\{X^N, N \geq 1\}$ if there exists a subsequence $\{N_k, k \geq 1\}$ of $\mathbb{N}$ such that the sequence $\{X^{N_k}, k \geq 1\}$ satisfies the large deviation principle with rate function I.

3.1.3 System model

We describe our model of the mean-field interacting particle system in a fast environment. Let there be N particles and an environment. There is a state associated with each particle as well as the environment at all times; the particle states come from a finite set $\mathcal{X}$ and the environment state comes from a finite set $\mathcal{Y}$. The state of the nth particle at time t is denoted by $X_n^N(t) \in \mathcal{X}$, and the state of the environment at time t is denoted by $Y^N(t) \in \mathcal{Y}$. To describe the evolution of the states of the particles, we consider a directed graph $(\mathcal{X}, \mathcal{E}_{\mathcal{X}})$ on the vertex set $\mathcal{X}$ with the interpretation that whenever $(x, x') \in \mathcal{E}_{\mathcal{X}}$, a particle at state x can transit to state x'. Similarly, to describe the evolution of the environment, we consider a directed graph $(\mathcal{Y}, \mathcal{E}_{\mathcal{Y}})$; $(y, y') \in \mathcal{E}_{\mathcal{Y}}$ implies that the environment can transit from state y to state y'.

To describe the particle transitions, we define, for each $y \in \mathcal{Y}$ and $(x, x') \in \mathcal{E}_{\mathcal{X}}$, a function $\lambda_{x,x'}(\cdot, y) : \mathcal{M}_1(\mathcal{X}) \to \mathbb{R}_+$, and for each $y \in \mathcal{Y}$, we consider the generator Q_y^N acting on functions on $\mathcal{X}^N$ by

$$Q_y^N f(\mathbf{x}^N) = \sum_{n=1}^{N} \sum_{x'_n : (x_n, x'_n) \in \mathcal{E}_{\mathcal{X}}} \lambda_{x_n, x'_n}(\overline{\mathbf{x}^N}, y)(f(\mathbf{x}^N_{n, x_n, x'_n}) - f(\mathbf{x}^N)),\ \mathbf{x}^N \in \mathcal{X}^N,$$

where $\overline{\mathbf{x}^N} := \frac{1}{N}\sum_{n=1}^N \delta_{x_n}$ denotes the empirical measure associated with the configuration $\mathbf{x}^N$, and $\mathbf{x}^N_{n,x_n,x'_n}$ denotes the resultant configuration of particles when the nth particles changes its state from x_n to x'_n in $\mathbf{x}^N$. To describe the transitions of the environment, for each $(y, y') \in \mathcal{Y}$, we define a function $\gamma_{y,y'}(\cdot) : \mathcal{M}_1(\mathcal{X}) \to \mathbb{R}_+$, and for each $\xi \in \mathcal{M}_1(\mathcal{X})$, we consider the generator L_ξ acting on functions on $\mathcal{Y}$ by

$$L_\xi g(y) = \sum_{y' : (y, y') \in \mathcal{E}_{\mathcal{Y}}} (g(y') - g(y))\gamma_{y,y'}(\xi),\ y \in \mathcal{Y}.$$

Finally, we consider the generator Ψ^N acting on functions f on $\mathcal{X}^N \times \mathcal{Y}$ by

$$\Psi^N f(\mathbf{x}^N, y) = Q_y^N f(\cdot, y)(\mathbf{x}^N) + N L_{\overline{\mathbf{x}^N}} f(\mathbf{x}^N, \cdot)(y),\ (\mathbf{x}^N, y) \in \mathcal{X}^N \times \mathcal{Y},$$

where $Q_y^N f(\cdot, y)(\mathbf{x}^N)$ (resp. $L_{\overline{\mathbf{x}^N}} f(\mathbf{x}^N, \cdot)(y)$) indicates that the operator Q_y^N (resp. $L_{\overline{\mathbf{x}^N}}$) acts on the first variable (resp. second variable) of f and the resultant function is evaluated at $\mathbf{x}^N$ (resp. y).

We make the following assumptions on the particle system:

(C1) The graph $(\mathcal{X}, \mathcal{E}_{\mathcal{X}})$ is irreducible;

(C2) For each $y \in \mathcal{Y}$ and $(x, x') \in \mathcal{E}_{\mathcal{X}}$, the function $\lambda_{x,x'}(\cdot, y)$ is Lipschitz continuous on $\mathcal{M}_1(\mathcal{X})$ and $\inf_{\xi \in \mathcal{M}_1(\mathcal{X})} \lambda_{x,x'}(\xi, y) > 0$;

and the following assumptions on the environment:

(D1) The graph $(\mathcal{Y}, \mathcal{E}_{\mathcal{Y}})$ is irreducible;

(D2) For each $(y, y') \in \mathcal{E}_{\mathcal{Y}}$, the function $\gamma_{y,y'}(\cdot)$ is continuous on $\mathcal{M}_1(\mathcal{X})$ and $\inf_{\xi \in \mathcal{M}_1(\mathcal{X})} \gamma_{y,y'}(\xi) > 0$.

As a consequence of the assumptions (C2) and (D2), we see that the transition rates of the particles as well as that of the environment are bounded, i.e.,

$$\sup_{\xi \in \mathcal{M}_1(\mathcal{X})} \lambda_{x,x'}(\xi, y) < +\infty\, \forall\, (x, x') \in \mathcal{E}_{\mathcal{X}} \text{ and } \forall\, y \in \mathcal{Y},$$

and

$$\sup_{\xi \in \mathcal{M}_1(\mathcal{X})} \gamma_{y,y'}(\xi) < +\infty\, \forall\, (y, y') \in \mathcal{E}_{\mathcal{Y}},$$

and hence the $D([0, \infty), \mathcal{X}^N \times \mathcal{Y})$-valued martingale problem for Ψ_N is well-posed (see, for example, Ethier and Kurtz [34, Section 4.1, Exercise 15]). Therefore, given an initial configuration of the particles $(X_n^N(0), 1 \le n \le N) \in \mathcal{X}^N$ and an initial state of the environment $Y^N(0) \in \mathcal{Y}$, we have a Markov process $\{((X_n^N(t), 1 \le n \le N), Y^N(t)), t \ge 0\}$ whose sample paths are elements of $D([0, \infty), \mathcal{X}^N \times \mathcal{Y})$.

To describe the process $\{((X_n^N(t), 1 \le n \le N), Y^N(t)), t \ge 0\}$ in words, consider the mapping

$$\{((X_n^N(t), 1 \le n \le N), Y^N(t)), t \ge 0\} \mapsto \left\{ \frac{1}{N} \sum_{n=1}^{N} \delta_{X_n^N(t)}, t \ge 0 \right\}$$
$$=: \{\mu^N(t), t \ge 0\} \in D([0,\infty), \mathcal{M}_1^N(\mathcal{X}))$$

that takes the process $\{((X_n^N(t), 1 \le n \le N), Y^N(t)), t \ge 0\}$ and maps it to the empirical measure process $\{\mu^N(t), t \ge 0\}$. Note that, if the environment were frozen to be y, then μ^N is Markov with infinitesimal generator

$$\Phi_y^N f(\xi) = \sum_{(x,x') \in \mathcal{E}_{\mathcal{X}}} N\xi(x)\lambda_{x,x'}(\xi, y) \left[f\left(\xi + \frac{\delta_{x'}}{N} - \frac{\delta_x}{N} \right) - f(\xi) \right], \, \xi \in \mathcal{M}_1^N(\mathcal{X}).$$

We see that a particle in state x at time t makes a transition to state x' at rate $\lambda_{x,x'}(\mu^N(t), Y^N(t))$ independent of everything else. Similarly, the environment makes a transition from state y to y' at time t at rate $N\gamma_{y,y'}(\mu^N(t))$ independent of everything else. Thus, the evolution of each particle depends on the empirical measure of the states of all the particles and the environment, and the evolution of the environment depends on the empirical measure of the states of all the particles. Note that the factor N in the second term of the generator Ψ^N indicates that the process Y^N makes $O(N)$ many transitions while each particle makes $O(1)$ transitions in a given $O(1)$ duration of time. Therefore, we have a "fully coupled" system where the particles evolve in a fast varying environment. Also, the empirical measure process μ^N makes $O(N)$ transitions over a given duration of time, but each of those transitions are of size $O(1/N)$ on the probability simplex $\mathcal{M}_1(\mathcal{X})$. We shall refer to μ^N as the slow process and Y^N as the fast process.

Remark 3.1. Throughout the chapter, we assume that all stochastic processes are defined on a complete filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \ge 0}, P)$. We denote integration with respect to P by $\mathcal{E}$.

Fix $T > 0$. We now describe the typical behaviour of our two time scale mean-field system for large N over the time duration $[0, T]$. Towards this, we define the occupation measure of the fast process Y^N by

$$\theta^N(t) := \int_0^t \mathbf{1}_{\{Y^N(s) \in \cdot\}} ds, \, t \in [0, T].$$

Note that $\theta^N \in D_{\uparrow}([0,T], \mathcal{M}(\mathcal{Y}))$, $\theta_t^N(\mathcal{Y}) = t$ and we can view θ^N as a measure on $[0,T] \times \mathcal{Y}$.

For a fixed empirical measure of the particles $\xi \in \mathcal{M}_1(\mathcal{X})$, assumptions (D1) and (D2) imply that there exists a unique invariant probability measure for the Markov process on $\mathcal{Y}$ with infinitesimal generator L_ξ (we denote this by π_ξ). Therefore, when the empirical measure at time t is at a fixed state μ_t, since the fast process Y^N makes $O(N)$ transitions, we expect that the occupation measure of Y^N for large N becomes "close" to π_{μ_t}, the unique invariant probability measure associated with L_{μ_t}. Due to this ergodic behaviour of the fast process, we anticipate that a particle in state x at time t moves to state x', where $(x,x') \in \mathcal{E}_\mathcal{X}$, at rate $\int_\mathcal{Y} \lambda_{x,x'}(\mu_t, y)\pi_{\mu_t}(dy)$, i.e., the average of $\lambda_{x,x'}(\mu_t, \cdot)$ over π_{μ_t} (for any $\xi \in \mathcal{M}_1(\mathcal{X})$, $(x,x') \in \mathcal{E}_\mathcal{X}$ and $m \in \mathcal{M}_1(\mathcal{Y})$, we define $\bar{\lambda}_{x,x'}(\xi, m) := \int_\mathcal{Y} \lambda_{x,x'}(\xi, y)m(dy)$).

More precisely, for large enough N, we anticipate the following averaging principle for the empirical measure process μ^N. If we assume that the initial conditions $\mu^N(0) \to \nu$ weakly for some deterministic element $\nu \in \mathcal{M}_1(\mathcal{X})$, then we anticipate that μ^N converges in probability, in $D([0,T], \mathcal{M}_1(\mathcal{X}))$, to the solution to the McKean-Vlasov ODE

$$\dot{\mu}_t = \bar{\Lambda}^*_{\mu_t, \pi_{\mu_t}} \mu_t,\ t \geq 0,\ \mu_0 = \nu, \tag{3.1}$$

where $\bar{\Lambda}_{\mu_t,\pi_{\mu_t}}$ denotes the $|\mathcal{X}|\times|\mathcal{X}|$ rate matrix of the slow process when the empirical measure is μ_t and the occupation measure of the fast process is π_{μ_t}, i.e., $\bar{\Lambda}_{\mu_t,\pi_{\mu_t}}(x,x') = \bar{\lambda}_{x,x'}(\mu_t, \pi_{\mu_t})$ when $(x,x') \in \mathcal{E}_\mathcal{X}$, $\bar{\Lambda}_{\mu_t,\pi_{\mu_t}}(x,x') = 0$ when $(x,x') \notin \mathcal{E}_\mathcal{X}$, $\bar{\Lambda}_{\mu_t,\pi_{\mu_t}}(x,x) = -\sum_{x' \neq x} \bar{\lambda}_{x,x'}(\mu_t, \pi_{\mu_t})$, and $\bar{\Lambda}^*_{\mu_t,\pi_{\mu_t}}$ denotes its transpose. Note that the above ODE is well-posed, thanks to the Lipschitz assumption on the transition rates (C2). See Bordenave et al. [14] for the study of averaging phenomena of a slightly general two time scale model in which each particle has a fast varying environment associated with it.

3.1.4 Main result

The result of this chapter is on the large deviations of $\{(\mu^N, \theta^N), N \geq 1\}$, the joint empirical measure process associated with the particle system and the occupation measure process associated with the environment Y^N, on $D([0,T], \mathcal{M}_1(\mathcal{X})) \times D_\uparrow([0,T], \mathcal{M}(\mathcal{Y}))$.

Theorem 3.1. *Assume (C1), (C2), (D1), (D2), and fix $T > 0$. Suppose that $\{\mu^N(0), N \geq 1\}$ satisfies the LDP on $\mathcal{M}_1(\mathcal{X})$ with rate function I_0. Then the sequence $\{((\mu^N(t), \theta^N(t)), t \in [0,T]), N \geq 1\}$ satisfies the LDP on $D([0,T], \mathcal{M}_1(\mathcal{X})) \times D_\uparrow([0,T], \mathcal{M}(\mathcal{Y}))$ with rate function*

$$I(\mu, \theta) := I_0(\mu(0)) + J(\mu, \theta),$$

where J is defined by

$$\begin{aligned} J(\mu,\theta) := \int_{[0,T]} \Bigg\{ &\sup_{\alpha\in\mathbb{R}^{\mathcal{X}}} \Bigg(\langle \alpha, (\dot{\mu}_t - \bar{\Lambda}^*_{\mu_t,m_t}\mu_t)\rangle \\ &\qquad - \int_{\mathcal{X}\times\mathcal{E}_{\mathcal{X}}} \tau(D\alpha(x,\Delta))\bar{\lambda}_{x,x+d\Delta}(\mu_t,m_t)\mu_t(dx)\Bigg) \\ &+ \sup_{g\in B(\mathcal{Y})} \int_{\mathcal{Y}} \Bigg(-L_{\mu_t} g(y) \\ &\qquad - \int_{\mathcal{E}_{\mathcal{Y}}} \tau(Dg(y,\Delta))\gamma_{y,y+d\Delta}(\mu_t)\Bigg) m_t(dy) \Bigg\}\, dt \end{aligned} \tag{3.2}$$

whenever the mapping $[0,T] \ni t \mapsto \mu_t \in \mathcal{M}_1(\mathcal{X})$ is absolutely continuous and θ, when viewed as a measure on $[0,T]\times\mathcal{Y}$, admits the representation $\theta(dydt) = m_t(dy)dt$ for some $m_t \in \mathcal{M}_1(\mathcal{Y})$ for almost all $t\in[0,T]$, and $J(\mu,\theta) = +\infty$ otherwise.

Note that our rate function consists of two parts – one corresponding to the empirical measure process μ^N and the other corresponding to the occupation measure of the fast process Y^N. The form of the first part of the rate function in (3.2) corresponding to the empirical measure process μ^N appears in the literature on large deviations of mean-field models (see Léonard [55, Theorem 3.3], Djehiche and Kaj [30, Theorem 1]). The form of the second part is related to the rate function that appears in the study of occupation measure of Markov processes (see Donsker and Varadhan [31, Theorem 1]). Here, the canonical form of the rate function is $\int_{[0,T]}\sup_{h>0}\int_{\mathcal{Y}} -\frac{L_{\mu_t}h(y)}{h(y)} m_t(dy)dt$ and this form of the second part of our rate function in (3.2) can be obtained by taking supremum over functions of the form e^g, $g\in B(\mathcal{Y})$. We see that the first part of the rate function corresponding to the empirical measure process μ^N has parameters of the mean-field model "averaged" by the fast variable. Further the second part corresponding to the occupation measure of the fast process has parameters "frozen" at the current value of the slow variable. The form of our rate function is similar in spirit to that obtained by Puhalskii [73] in the case of coupled diffusions.

Note that, when μ is the solution to the McKean-Vlasov equation (3.1) starting at $\mu(0)$ and θ, when viewed as a measure on $[0,T]\times\mathcal{Y}$, is given by $\theta(dydt) = \pi_{\mu_t}(dy)dt$ where π_{μ_t} is the unique invariant probability measure associated with the infinitesimal generator L_{μ_t}, it is easy to see that the suprema in (3.2) are attained at the identically 0 functions $\alpha\equiv 0$ and $g\equiv 0$ and hence $J(\mu,\theta)=0$. Therefore, we recover the typical behaviour of our fully coupled system – at each time $t>0$, the empirical measure process μ^N tracks the solution to the McKean-Vlasov equation μ_t starting at $\mu(0)$ and the occupation measure of the fast process θ^N tracks the invariant probability measure of the fast process Y^N when the empirical measure is frozen at

μ_t. Our result on the large deviations of the joint empirical measure process and the occupation measure of the fast process $\{(\mu^N, \theta^N)\}$ enables us to estimate the probabilities of two kinds of deviations from the typical behaviour – one where, for a given μ, the occupation measure of the fast process deviates from its typical behaviour (which at time t is $\pi_{\mu_t}(dy)dt$) and the other where μ deviates from its typical behaviour (which is the solution to (3.1) starting at $\mu(0)$).

We now provide an outline of the proof of Theorem 3.1. Our proof is broadly built upon the methodology of stochastic exponentials for large deviations by Puhalskii [70, 71, 73], where one shows the large deviation principle by first obtaining an equation for a subsequential rate function in terms of a suitable exponential martingale and then obtaining a characterisation of this subsequential rate function. Towards this, we first show that the sequence $\{(\mu^N, \theta^N), N \geq 1\}$ is exponentially tight in $D([0,T], \mathcal{M}_1(\mathcal{X})) \times D_\uparrow([0,T], \mathcal{M}(\mathcal{Y}))$ (see Theorem 3.4); this is shown using standard martingale arguments and Doob's inequality. Exponential tightness of the sequence $\{(\mu^N, \theta^N), N \geq 1\}$ implies that there exists a subsequence $\{N_k, k \geq 1\}$ of $\mathbb{N}$ such that the family $\{(\mu^{N_k}, \theta^{N_k}), k \geq 1\}$ satisfies the LDP (see, for example, Dembo and Zeitouni [29, Lemma 4.1.23]); let $\tilde{I}$ denote the rate function that governs the LDP for the family $\{(\mu^{N_k}, \theta^{N_k}), k \geq 1\}$. In Sections 3.3-3.6, we obtain a characterisation of $\tilde{I}$ when $\tilde{I}$ is such that, for some $\nu \in \mathcal{M}_1(\mathcal{X})$, $\tilde{I}(\mu, \theta) = +\infty$ unless $\mu_0 = \nu$; specifically we show that $\tilde{I}(\mu, \theta)$ is given by the right hand side of (3.2). In some more detail, in Section 3.3, we define an exponential martingale associated with the Markov process (μ^N, Y^N) for a class of functions $\alpha : [0,T] \times \mathcal{M}_1(\mathcal{X}) \to \mathbb{R}^{\mathcal{X}}$ and $g : [0,T] \times \mathcal{M}_1(\mathcal{X}) \times \mathcal{Y} \to \mathbb{R}$ with certain properties, and we obtain an equation that the rate function $\tilde{I}$ must satisfy in terms of this exponential martingale (see Theorem 3.5). In Section 3.4, we define our candidate rate function I^* in terms of this exponential martingale as a variational problem over functions α and g, and we then show that I^* coincides with the RHS of (3.2), and provide a nonvariational expression for I^* using elements from suitable Orlicz spaces (see Theorem 3.6). In Section 3.5, using the properties of the solution to the variational problem established in Section 3.4 and an extension of the equation of $\tilde{I}$ to a larger class of functions α and g, we are able to obtain a characterisation of the rate function $\tilde{I}$ for sufficiently regular elements in $D([0,T], \mathcal{M}_1(\mathcal{X})) \times D_\uparrow([0,T], \mathcal{M}(\mathcal{Y}))$ (see Theorem 3.8). In Section 3.6, we extend the above characterisation of $\tilde{I}$ to the whole space $D([0,T], \mathcal{M}_1(\mathcal{X})) \times D_\uparrow([0,T], \mathcal{M}(\mathcal{Y}))$ via certain approximation arguments. We finally complete the proof of Theorem 3.1 in Section 3.7, by removing the restriction that, for some $\nu \in \mathcal{M}_1(\mathcal{X})$, $\tilde{I}(\mu, \theta) = +\infty$ unless $\mu_0 = \nu$.

Our setting of mean-field interaction with jumps introduces some difficulties in characterising a subsequential rate function. One of them is in obtaining regularity properties of the solution to the variational problem appearing in the definition of $J(\mu, \theta)$ in (3.2) when (μ, θ) possesses some good properties. In the recent work of Puhalskii [73] on large deviations of fully coupled diffusions, the author uses tools from the theory of elliptic partial differential equations for this purpose whereas we resort to tools from convex analysis (Léonard [56, Sections 4-6]) and parametric continuity of optimisation problems (Sundaram [82, Chapter 9]) – see Theorem 3.6 and Theorem 3.8. Also, unlike in the case of Gaussian noise in Puhalskii [73], our Poissonian noise prevents us from obtaining an explicit form of the solution to the variational problem appearing in the rate function (3.2). Yet another difficulty is in obtaining a characterisation of $\tilde{I}(\mu,\theta)$ when the path μ hits the boundary of $\mathcal{M}_1(\mathcal{X})$. In such cases, the solution to the variational problem that appears in (3.2) blows up near the boundary and hence the condition on $\tilde{I}$ established in Theorem 3.7 cannot be directly used. We demonstrate how to approximate (μ,θ) via a sequence of regular elements $\{(\mu^i,\theta^i)\}_{i\geq 1}$ so that the solution to the variational problem in $J(\mu^i,\theta^i)$ is well-behaved. We can then use the conclusion of Theorem 3.7 on the above sequence and show that $\tilde{I}(\mu^i,\theta^i)\to\tilde{I}(\mu,\theta)$ as $i\to\infty$; see Theorem 3.9.

3.1.4.1 Marginal μ^N

The above result on large deviations of the joint law of the empirical measure process of the particles and the occupation measure of the fast process enables us to easily obtain large deviations of the empirical measure process μ^N by using the contraction principle (see, for example, Dembo and Zeitouni [29, Theorem 4.2.1]).

Corollary 3.1. *Assume (C1), (C2), (D1), (D2), and fix $T>0$. Suppose that $\{\mu^N(0), N\geq 1\}$ satisfies the LDP on $\mathcal{M}_1(\mathcal{X})$ with rate function I_0. Then $\{\mu^N, N\geq 1\}$ satisfies the LDP on $D([0,T],\mathcal{M}_1(\mathcal{X}))$ with rate function J_T defined as follows. If $[0,T]\ni t\mapsto\mu_t$ is absolutely continuous, then*

$$J_T(\mu)=I_0(\mu_0)+\int_{[0,T]}\Bigg\{\sup_{\alpha\in\mathbb{R}^{\mathcal{X}}}\Bigg(\langle\alpha,\dot{\mu}_t\rangle-\sup_{m\in\mathcal{M}_1(\mathcal{Y})}\Bigg[\langle\alpha,\bar{\Lambda}^*_{\mu_t,m}\mu_t\rangle + \int_{\mathcal{X}\times\mathcal{E}_{\mathcal{X}}}\tau(D\alpha(x,\Delta))\bar{\lambda}_{x,x+d\Delta}(\mu_t,m)\mu_t(dx)\Bigg) - \sup_{g\in B(\mathcal{Y})}\int_{\mathcal{Y}}\Bigg(-L_{\mu_t}g(y)-\int_{\mathcal{E}_{\mathcal{Y}}}\tau(Dg(y,\Delta))\gamma_{y,y+d\Delta}(\mu_t)\Bigg)m(dy)\Bigg]\Bigg\}\,dt,$$

where θ, when viewed as a measure on $[0,T]\times\mathcal{Y}$, admits the representation $\theta(dydt)=m_t(dy)dt$ for some $m_t\in\mathcal{M}_1(\mathcal{Y})$ for almost all $t\in[0,T]$, and $J_T(\mu)=+\infty$ otherwise.

3.1.4.2 Large time behaviour

Using the result on the finite duration LDP for the process $\{\mu^N, N \geq 1\}$ in Corollary 3.1, we can employ the tools of Freidlin and Wentzell [37, Chapter 6] and Hwang and Sheu [44] to study the large time behaviour of the process μ^N. The programme to understand the large time behaviour is carried out in Section 2.3. The two crucial properties needed to establish the large time behaviour of μ^N are: (i) the continuity of the Freidlin-Wentzell quasipotential (see Section 2.3 for its definition) and (ii) uniform large deviations of μ^N, uniformly with respect to the initial condition $\mu^N(0)$ lying in a given closed set. One can show that the Freidlin-Wentzell quasipotential is continuous on $\mathcal{M}_1(\mathcal{X}) \times \mathcal{M}_1(\mathcal{X})$ by constructing constant velocity trajectories between any two given points in $\mathcal{M}_1(\mathcal{X})$ and estimating the corresponding J_T for that path; see Borkar and Sundaresan [15, Lemma 3.4]. Since the space $\mathcal{M}_1(\mathcal{X})$ is compact, one can also establish uniform large deviation estimates, see Corollary 2.1. Using the above two properties and the fact that (μ^N, Y^N) is strong Markov, one can establish results on the large time behaviour of μ^N such as (i) the mean exit time from a neighbourhood of an ω-limit set of (3.1), (ii) the probability of reaching a given ω-limit set starting from another, etc.

3.2 Exponential tightness

In this section, we prove the exponential tightness of the sequence $\{((\mu^N(t), \theta^N(t)), t \in [0,T]), N \geq 1\}$ in $D([0,T], \mathcal{M}_1(\mathcal{X})) \times D_{\uparrow}([0,T], \mathcal{M}(\mathcal{Y}))$. Towards this, we shall use the following results (Theorems 3.2-3.3). The proof of these results are standard and will be omitted here (see Feng and Kurtz [36, Theorem 4.4] and Puhalskii [70, Theorem B]).

Theorem 3.2. *A sequence $\{X^N\} = \{X_t^N, t \in [0,T]\}$ taking values in $D([0,T], S)$ is exponentially tight if and only if*

(i) for each $M > 0$, there exists a compact set $K_M \subset S$ such that

$$\limsup_{N \to \infty} \frac{1}{N} \log P(\exists t \in [0,T] \text{ such that } X_t^N \notin K_M) \leq -M,$$

(ii) there exists a family of functions $F \subset C(S)$ that is closed under addition and separates points on S such that for each $f \in F$, $\{f(X^N)\}$ is exponentially tight in $D([0,T], \mathbb{R})$.

See Feng and Kurtz [36, Theorem 4.4] for a proof. We also need the following sufficient condition for exponential tightness in $D([0,T], \mathbb{R})$.

Theorem 3.3. *Let $\{X^N\}$ be a sequence taking values in $D([0,T],\mathbb{R})$. Suppose that*

(i) we have

$$\lim_{M\to\infty}\limsup_{N\to\infty}\frac{1}{N}\log P(\exists t\in[0,T] \text{ such that } |X_t^N|>M)=-\infty,$$

(ii) for each $\varepsilon>0$,

$$\lim_{\delta\downarrow 0}\limsup_{N\to\infty}\frac{1}{N}\log\sup_{t_1\in[0,T]}P(\sup_{t_2\in[t_1,t_1+\delta]}|X_{t_2}^N-X_{t_1}^N|>\varepsilon)=-\infty.$$

Then $\{X^N\}$ is exponentially tight in $D([0,T],\mathbb{R})$.

See Puhalskii [70, Theorem B] for a proof.

We now show the main result of this section, namely exponential tightness of the sequence $\{(\mu^N,\theta^N),N\geq 1\}$.

Theorem 3.4. *The sequence of random variables $\{((\mu^N(t),\theta^N(t)),t\in[0,T]),N\geq 1\}$ is exponentially tight in $D([0,T],\mathcal{M}_1(\mathcal{X}))\times D_{\uparrow}([0,T],\mathcal{M}(\mathcal{Y}))$, i.e., given any $M>0$, there exists a compact set $K_M\subset D([0,T],\mathcal{M}_1(\mathcal{X}))\times D_{\uparrow}([0,T],\mathcal{M}(\mathcal{Y}))$ such that*

$$\limsup_{N\to\infty}\frac{1}{N}\log P\left(\{(\mu^N(t),\theta^N(t)),t\in[0,T]\}\notin K_M\right)\leq -M.$$

Proof. It suffices to show that μ^N and θ^N are individually exponentially tight in $D([0,T],\mathcal{M}_1(\mathcal{X}))$ and $D_{\uparrow}([0,T],\mathcal{M}(\mathcal{Y}))$ respectively (see, for example, Feng and Kurtz [36, Lemma 3.6]).

Consider θ^N. Note that, for $t\in[0,T]$, we have $|\theta_t^N(Y)|\leq t$ for any subset $Y\subset\mathcal{Y}$. Therefore, using the compact set $K_M=\{y\in\mathbb{R}^{\mathcal{Y}}:0\leq y_i\leq T\,\forall i\}\subset\mathcal{M}(\mathcal{Y})$, condition (i) of Theorem 3.2 holds. To verify condition (ii), define the collection of functions $F:=\{f:\mathcal{M}(\mathcal{Y})\to\mathbb{R}:f(\theta)=\langle\alpha,\theta\rangle,\alpha\in\mathbb{R}^{\mathcal{Y}}\}$. Clearly, F is closed under addition and separates points on $\mathcal{M}(\mathcal{Y})$. For any f of the form $f(\theta)=\langle\alpha,\theta\rangle$ for some $\alpha\in\mathbb{R}^{\mathcal{Y}}$, note that, with $X_t^N=f(X_t^N)$, condition (i) of Theorem 3.3 holds since $|X_t^N|\leq t\max_{i\in\mathcal{Y}}|\alpha_i|$. To verify condition (ii) of Theorem 3.3, note that, for any $0\leq s\leq t\leq T$, we have $|\theta_t^N(Y)-\theta_s^N(Y)|\leq t-s$ for any $Y\subset\mathcal{Y}$ and hence $|X_t^N-X_s^N|\leq(t-s)\max_i|\alpha_i|$. Thus, by choosing a sufficiently small $\delta>0$, it is easy to see that condition (ii) of Theorem 3.3 holds. This establishes the exponential tightness of $\{\theta^N\}$ in $D_{\uparrow}([0,T],\mathcal{M}(\mathcal{Y}))$.

We now show that μ^N is exponentially tight in $D([0,T],\mathcal{M}_1(\mathcal{X}))$. Since for each $t>0$, μ_t^N takes values in a compact space, condition (i) of Theorem 3.2 holds trivially. Again, to show condition (ii) of Theorem 3.2, we shall make use of Theorem 3.3. For this, we fix the class of functions $F := \{f : \mathcal{M}_1(\mathcal{X}) \to \mathbb{R}_+ : f(\xi) = \langle \alpha, \xi\rangle, \alpha \in \mathbb{R}^{\mathcal{X}}\}$, which is clearly closed under addition and separates points on $\mathcal{M}_1(\mathcal{X})$. Fix $f \in F$ such that $f(\xi) = \langle \alpha, \xi\rangle$ for some $\alpha \in \mathbb{R}^{\mathcal{X}}$ and let $X_t^N = f(\mu_t^N) = \langle \alpha, \mu_t^N\rangle$. Note that, we have $|X_t^N| \leq \max_x |\alpha_x|$ for all $t \geq 0$ and $N \geq 1$, hence condition (i) of Theorem 3.3 holds. To check condition (ii), note that, for each $t_1 \geq 0$ and $\beta > 1$,

$$\begin{aligned}
M_t := \exp\Bigg\{N\Bigg(\beta X_t^N - \beta X_{t_1}^N - \beta\int_{t_1}^t \Phi_{Y_s^N} f(\mu_s^N)ds \\
- \int_{t_1}^t \int_{\mathcal{X}\times\mathcal{E}_{\mathcal{X}}} \tau(\beta D\alpha(x,\Delta))\lambda_{x,x+d\Delta}(\mu_s^N, Y_s^N)\mu_s^N(dx)ds\Bigg)\Bigg\}, t \geq t_1,
\end{aligned}$$

is an $\mathcal{F}_t$-martingale (see Léonard [56, Lemma 3.3]; alternatively, this can be easily checked using the Doléans-Dade exponential formula, see, for example, Jacod and Shiryaev [45, Chapter I, Theorem 4.61]). Therefore, given $\varepsilon > 0$, $\delta > 0$ and $t_1 > 0$, we have

$$\begin{aligned}
&P\left(\sup_{t_2\in[t_1,t_1+\delta]} (X_{t_2}^N - X_{t_1}^N) > \varepsilon\right) \\
&= P\left(\sup_{t_2\in[t_1,t_1+\delta]} \exp\{N\beta(X_{t_2}^N - X_{t_1}^N)\} > \exp\{N\beta\varepsilon\}\right) \\
&= P\Bigg(\sup_{t\in[t_1,t_1+\delta]} M_t \times \exp\Bigg\{N\beta\int_{t_1}^t \Phi_{Y_s^N} f(\mu_s^N)ds \\
&\qquad +N\int_{t_1}^t\int_{\mathcal{X}\times\mathcal{E}_{\mathcal{X}}} \tau(\beta D\alpha(x,\Delta))\lambda_{x,x+d\Delta}(\mu_s^N, Y_s^N)\mu_s^N(dx)ds\Bigg\} > \exp\{N\beta\varepsilon\}\Bigg) \\
&\leq P\left(\sup_{t\in[t_1,t_1+\delta]} M_t \exp\{N\delta c_{\alpha,\beta}\} > \exp\{N\beta\varepsilon\}\right) \\
&\leq \exp\{-N(\beta\varepsilon - \delta c_{\alpha,\beta})\}
\end{aligned}$$

where $c_{\alpha,\beta}$ is a constant depending on α and β; here the first inequality follows from the boundedness of the transition rates which is a consequence of the Lipschitz assumption (C2), and the second inequality follows from Doob's martingale inequality and the fact that $EM_t =$

$EM_{t_1} = 1$ for all $t \geq t_1$. Thus, we obtain

$$\lim_{\delta \downarrow 0} \limsup_{N\to\infty} \frac{1}{N} \log \sup_{t_1\in[0,T]} P\left(\sup_{t_2\in[t_1,t_1+\delta]} (X^N_{t_2} - X^N_{t_1}) > \varepsilon \right) \leq -\beta\varepsilon,$$

and hence, letting $\beta \to \infty$, we have

$$\lim_{\delta \downarrow 0} \limsup_{N\to\infty} \frac{1}{N} \log \sup_{t_1\in[0,T]} P\left(\sup_{t_2\in[t_1,t_1+\delta]} (X^N_{t_2} - X^N_{t_1}) > \varepsilon \right) = -\infty.$$

We can now replace α with $-\alpha$ and repeat the above arguments to conclude that

$$\lim_{\delta \downarrow 0} \limsup_{N\to\infty} \frac{1}{N} \log \sup_{t_1\in[0,T]} P\left(\sup_{t_2\in[t_1,t_1+\delta]} |X^N_{t_2} - X^N_{t_1}| > \varepsilon \right) = -\infty.$$

We have thus verified condition (ii) of Theorem 3.3 and hence it follows that $\{\mu^N, N \geq 1\}$ is exponentially tight in $D([0,T], \mathcal{M}_1(\mathcal{X}))$. This completes the proof of the theorem. □

3.3 An equation for the subsequential rate function

Let $\tilde{I} : D([0,T], \mathcal{M}_1(\mathcal{X})) \times D_\uparrow([0,T], \mathcal{M}(\mathcal{Y})) \to [0, +\infty]$ denote a subsequential rate function for the family $\{(\mu^N, \theta^N), N \geq 1\}$, i.e., for some sequence $\{N_k, k \geq 1\}$ of $\mathbb{N}$, the family $\{(\mu^{N_k}, \theta^{N_k}), k \geq 1\}$ satisfies the large deviation principle with rate function $\tilde{I}$. In this section, we obtain a condition that every such subsequential rate function must satisfy.

We start with some definitions. Given $g \in C^{1,1}([0,T] \times \mathcal{M}_1(\mathcal{X}) \times \mathcal{Y})$, define

$$\begin{aligned}
V^g_t(\mu^N, Y^N) &\coloneqq g_t(\mu^N(t), Y^N(t)) - g_0(\mu^N(0), Y^N(0)) - \int_0^t \frac{\partial g_s}{\partial s}(\mu^N(s), Y^N(s)) ds \\
&\quad - \int_0^t \sum_{(x,x')\in\mathcal{E}_\mathcal{X}} \left[g_s\left(\mu^N(s) + \frac{\delta_{x'} - \delta_x}{N}, Y^N(s)\right) \right. \\
&\qquad \left. - g_s(\mu^N(s), Y^N(s)) \right] \times N\mu^N_s(x)\lambda_{x,x'}(\mu^N(s), Y^N(s)) ds \\
&\quad - \int_0^t \sum_{(x,x')\in\mathcal{E}_\mathcal{X}} \tau\left(\left[g_s\left(\mu^N(s) + \frac{\delta_{x'} - \delta_x}{N}, Y^N(s)\right) \right.\right. \\
&\qquad \left.\left. - g_s(\mu^N(s), Y^N(s)) \right] \right) \times N\mu^N_s(x)\lambda_{x,x'}(\mu^N(s), Y^N(s)) ds.
\end{aligned}$$

$$(3.3)$$

Let $n \in \mathbb{N}$. Given the time points $0 = t_0 < t_1 < \cdots < t_n = T$, $\alpha = (\alpha_{t_i})_{i=0}^{n-1}$ where $\alpha_{t_i} : \mathcal{M}_1(\mathcal{X}) \to \mathbb{R}^{\mathcal{X}}$ is continuous for each $0 \le i \le n-1$, and $\mu \in D([0,T], \mathcal{M}_1(\mathcal{X}))$, define

$$\int_0^t \alpha_s(\mu_s) d\mu_s := \sum_{i=1}^{n} \langle \alpha_{t_{i-1}}(\mu_{t_{i-1}}), (\mu_{t\wedge t_i} - \mu_{t \wedge t_{i-1}}) \rangle, t \in [0,T]; \tag{3.4}$$

note that this object is an element of $D([0,T],\mathbb{R})$. Given $s \in [0,T]$, define

$$\alpha_s(\mu_s) := \sum_{i=1}^{n} \alpha_{t_{i-1}}(\mu_{t_{i-1}}) \mathbf{1}_{\{s \in [t_{i-1}, t_i)\}}.$$

Then, given $s \in [0,T]$, $x \in \mathcal{X}$, and $\Delta = (x,x') \in \mathcal{E}_{\mathcal{X}}$, we have

$$D\alpha_s(\mu_s)(x,\Delta) = \sum_{i=1}^{n} (\alpha_{t_{i-1}}(\mu_{t_{t-1}})(x') - \alpha_{t_{i-1}}(\mu_{t_{i-1}})(x)) \mathbf{1}_{\{t \in [t_{i-1}, t_i)\}}.$$

Similarly, given $s \in [0,T]$, $y \in \mathcal{Y}$, and $\Delta = (y,y') \in \mathcal{E}_{\mathcal{Y}}$, we have

$$Dg_s(\mu_s, y, \Delta) = g_s(\mu_s, y') - g_s(\mu_s, y).$$

Finally, given $(\mu, \theta) \in D([0,T], \mathcal{M}_1(\mathcal{X})) \times D_{\uparrow}([0,T], \mathcal{M}(\mathcal{Y}))$, time points $0 = t_0 < t_1 < \cdots < t_n = T$, $\alpha = (\alpha_{t_i})_{i=0}^{n-1}$ and g that satisfy the above requirements, define

$$\begin{aligned} U_t^{\alpha,g}(\mu,\theta) := &\int_0^t \alpha_s(\mu_s) d\mu_s - \int_0^t \Big\langle \alpha_s(\mu_s), \int_{\mathcal{Y}} \Lambda^*_{\mu_s,y} \mu_s m_s(dy) \Big\rangle ds \\ &- \int_0^t \int_{\mathcal{X} \times \mathcal{E}_{\mathcal{X}} \times \mathcal{Y}} \tau(D\alpha_s(\mu_s)(x,\Delta)) \lambda_{x,x+d\Delta}(\mu_s, y) \mu_s(dx) m_s(dy) ds \\ &- \int_0^t \int_{\mathcal{Y}} \Big(L_{\mu_s} g_s(\mu_s, \cdot)(y) \\ &\quad + \int_{\mathcal{E}_{\mathcal{Y}}} \tau(Dg_s(\mu_s, y, \Delta)) \gamma_{y,y+d\Delta}(\mu_s) \Big) m_s(dy) ds; \end{aligned} \tag{3.5}$$

here θ, when viewed as a measure on $[0,T] \times \mathcal{Y}$, admits the representation $\theta(dydt) = m_t(dy)dt$ for some $m_t \in \mathcal{M}_1(\mathcal{Y})$ for almost all $t \in [0,T]$, which follows from the existence of the regular conditional distribution (see, for example, Ethier and Kurtz [34, Theorem 8.1, page 502]).

We prove the following result, a condition that $\tilde{I}$ must satisfy in terms of the functions $U^{\alpha,g}$.

Theorem 3.5. *Let $\tilde{I} : D([0,T], \mathcal{M}_1(\mathcal{X})) \times D_{\uparrow}([0,T], \mathcal{M}(\mathcal{Y})) \to [0,+\infty]$ denote a rate function and suppose that there is a subsequence $\{(\mu^{N_k}, \theta^{N_k}), k \geq 1\}$ of $\{(\mu^N, \theta^N), N \geq 1\}$ that satisfies the LDP with rate function $\tilde{I}$. Then, for each α and g that satisfy the requirements of the definition of U and V in (3.5) and (3.3) respectively, we have*

$$\sup_{(\mu,\theta)\in D([0,T],\mathcal{M}_1(\mathcal{X}))\times D_{\uparrow}([0,T],\mathcal{M}(\mathcal{Y}))} (U_T^{\alpha,g}(\mu,\theta) - \tilde{I}(\mu,\theta)) = 0. \tag{3.6}$$

Proof. Note that, since the transition rates are bounded (which is a consequence of the assumptions (C2) and (D2)),

$$N\left(\int_0^t \alpha_s(\mu_s)d\mu_s^N - \int_0^t \left\langle \alpha_s(\mu_s), \int_{\mathcal{Y}} \Lambda^*_{\mu_s^N,y}\mu_s^N\theta^N(dyds)\right\rangle\right), t \geq 0,$$

is an $\mathcal{F}_t$-martingale. Also, by Itô's formula,

$$\begin{aligned} g_t(\mu^N(t), Y^N(t)) - g_0(\mu^N(0), Y^N(0)) - \int_0^t \frac{\partial g_s}{\partial s}(\mu^N(s), Y^N(s))ds \\ - \int_0^t \sum_{(x,x')\in\mathcal{E}_{\mathcal{X}}} \Bigg[g_s\left(\mu^N(s) + \frac{\delta_{x'}-\delta_x}{N}, Y^N(s)\right) \\ - g_s(\mu^N(s), Y^N(s))\Bigg] \times N\mu_s^N(x)\lambda_{x,x'}(\mu^N(s), Y^N(s))ds \\ - N\int_0^t L_{\mu^N(s)}g_s(\mu^N(s),\cdot)(Y^N(s))ds, t \geq 0, \end{aligned}$$

is an $\mathcal{F}_t$-martingale. Therefore, using the Doléans-Dade exponential formula, it follows that

$$\exp\{NU_t^{\alpha,g}(\mu^N,\theta^N) + V_t^g(\mu^N, Y^N)\}, t \geq 0,$$

is an $\mathcal{F}_t$-martingale, and hence

$$E\exp\{NU_T^{\alpha,g}(\mu^N,\theta^N) + V_T^g(\mu^N, Y^N)\} = 1.$$

Clearly, $U_T^{\alpha,g}(\cdot,\cdot)$ is continuous on $D([0,T], \mathcal{M}_1(\mathcal{X})) \times D_{\uparrow}([0,T], \mathcal{M}(\mathcal{Y}))$, and since g is continuously differentiable in the second argument, $V_T^g(\mu^N, Y^N)$ is bounded, and hence $V_T^g(\mu^N, Y^N)/N$ goes to 0 P-a.s. Therefore, the result follows from an application of Varadhan's lemma along the subsequence $\{N_k, k \geq 1\}$ (see, for example, [29, Theorem 4.3.1]). □

3.4 The variational problem in J

Motivated by the duality relation (3.6), we define our candidate rate function

$$I^*(\mu,\theta) := \sup_{\alpha,g} U_T^{\alpha;g}(\mu,\theta), \tag{3.7}$$

where the supremum is taken over all functions α and g that satisfy the conditions in Theorem 3.5.

In this section, we study the above variational problem and show that, whenever $I^*(\mu,\theta) < +\infty$, $I^*(\mu,\theta)$ coincides with the RHS of (3.2) and that $I^*(\mu,\theta)$ can be expressed in a non-variational form using elements from suitable Orlicz spaces. We begin with a necessary condition on the elements of $D([0,T],\mathcal{M}_1(\mathcal{X})) \times D_{\uparrow}([0,T],\mathcal{M}(\mathcal{Y}))$ whose I^* is finite.

Lemma 3.1. *If $I^*(\mu,\theta) < +\infty$, then the mapping $[0,T] \ni t \mapsto \mu_t \in \mathcal{M}_1(\mathcal{X})$ is absolutely continuous.*

Proof. Take $g \equiv 0$ and α to be a function of only time (and denote this by α_t) in the definition of $U_t^{\alpha,g}$ in (3.5). Then (3.7) becomes

$$\begin{aligned} I^*(\mu,\theta) &= \sup_{\alpha,g} U_T^{\alpha;g}(\mu,\theta) \\ &\geq \int_0^T \alpha_t d\mu_t - \int_0^T \langle \alpha_t, \bar{\Lambda}^*_{\mu_t,m_t}\mu_t\rangle dt \\ &\qquad - \int_0^T \int_{\mathcal{X}\times\mathcal{E}_{\mathcal{X}}} \tau(D\alpha_t(x,\Delta))\bar{\lambda}_{x,x+d\Delta}(\mu_t,m_t)\mu_t(dx)dt. \end{aligned}$$

Therefore,

$$\begin{aligned} \int_0^T \alpha_t d\mu_t \leq I^*(\mu,\theta) &+ \int_0^T \langle \alpha_t, \bar{\Lambda}^*_{\mu_t,m_t}\mu_t\rangle dt \\ &+ \int_0^T \int_{\mathcal{X}\times\mathcal{E}_{\mathcal{X}}} \tau(D\alpha_t(x,\Delta))\bar{\lambda}_{x,x+d\Delta}(\mu_t,m_t)\mu_t(dx)dt. \end{aligned}$$

Replacing $c\alpha_t$ in place of α_t in the above equation, dividing throughout by c and choosing $c = 1/\|D\alpha\|_{L^\tau([0,T]\times\mathcal{X}\times\mathcal{E}_{\mathcal{X}},\bar{\lambda}_{x,x+d\Delta}(\mu_t,m_t)\mu_t(dx)dt)}$ (i.e. the inverse of the norm of the function $D\alpha$ in the Orlicz space $L^\tau([0,T]\times\mathcal{X}\times\mathcal{E}_\mathcal{X},\bar\lambda_{x,x+d\Delta}(\mu_t,m_t)\mu_t(dx)dt))$, we have

$$\int_0^T \alpha_t d\mu_t \le \|D\alpha\|_{L^\tau([0,T]\times\mathcal{X}\times\mathcal{E}_\mathcal{X},\bar\lambda_{x,x+d\Delta}(\mu_t,m_t)\mu_t(dx)dt)}(I^*(\mu,\theta)+1) + \int_0^T \langle\alpha_t,\bar\Lambda^*_{\mu_t,m_t}\mu_t\rangle dt.$$

Since α is arbitrary, from the definition of $\int_0^t \alpha_t d\mu_t$ in (3.4), it is clear that the mapping $[0,T]\ni t\mapsto\mu_t\in\mathcal{M}_1(\mathcal{X})$ is absolutely continuous. □

We also need the following lemma, whose proof can be found in Puhalskii [71, Lemma A.2, page 460].

Lemma 3.2. *Let $\mathcal{V}$ be a complete separable metric space, and let $\mathcal{U}$ be a dense subspace of $\mathcal{V}$. Let $f(t,v)$ be a function defined on $[0,T]\times\mathcal{V}$ that is measurable in t and continuous in v. Further, if $f(t,\beta(t))$ is locally integrable with respect to the Lebesgue measure on $[0,T]$ for all measurable functions $\beta:[0,T]\to\mathcal{U}$, then*

$$\sup_{\beta(\cdot)}\int_0^T f(t,\beta(t))dt = \int_0^T \sup_{y\in\mathcal{U}} f(t,y)dt,$$

where the supremum in the LHS is taken over all $\mathcal{U}$-valued measurable functions $\beta(\cdot)$.

Let us introduce some notations. Let $D\mathcal{C}_\mathcal{X}$ (resp. $D\mathcal{C}_\mathcal{Y}$) denote the space of functions $D\alpha$ (resp. Dg) on $[0,T]\times\mathcal{X}\times\mathcal{E}_\mathcal{X}$ (resp. $[0,T]\times\mathcal{Y}\times\mathcal{E}_\mathcal{Y}$) such that $\alpha\in C^1([0,T]\times\mathcal{X})$ (resp. $g\in C^1([0,T]\times\mathcal{Y})$). (For economy of notation in the sequel, we shall also view $\mathbb{R}$-valued functions on $[0,T]\times\mathcal{X}$ as $\mathbb{R}^\mathcal{X}$-valued functions on $[0,T]$.) Given $(\mu,\theta)\in D([0,T],\mathcal{M}_1(\mathcal{X}))\times D_\uparrow([0,T],\mathcal{M}(\mathcal{Y}))$, let $\mathcal{H}_\mathcal{X}(\mu,\theta)$ denote the $L^{\tau^*}([0,T]\times\mathcal{X}\times\mathcal{E}_\mathcal{X},\bar\lambda_{x,x+d\Delta}(\mu_t,m_t)\mu_t(dx)dt)$-closure of functions of the form $\{\exp\{D\alpha\}-1, D\alpha\in D\mathcal{C}_\mathcal{X}\}$ and let $\mathcal{H}_\mathcal{Y}(\mu,\theta)$ denote the $L^{\tau^*}([0,T]\times\mathcal{Y}\times\mathcal{E}_\mathcal{Y},\gamma_{y,y+d\Delta}(\mu_t)m_t(dy)dt)$-closure of functions of the form $\{\exp\{Dg\}-1, Dg\in D\mathcal{C}_\mathcal{Y}\}$, where θ admits the representation $\theta(dydt)=m_t(dy)dt$ for some $m_t\in\mathcal{M}_1(\mathcal{Y})$ for almost all $t\in[0,T]$. We now prove the main result of this section.

Theorem 3.6. *Suppose that $(\mu,\theta)\in D([0,T],\mathcal{M}_1(\mathcal{X}))\times D_\uparrow([0,T],\mathcal{M}(\mathcal{Y}))$ is such that $I^*(\mu,\theta)<$*

∞. *Then, we have*

$$I^*(\mu,\theta) = \int_{[0,T]} \Bigg\{ \sup_{\alpha\in\mathbb{R}^{\mathcal{X}}} \Bigg(\left\langle \alpha, (\dot{\mu}_t - \bar{\Lambda}^*_{\mu_t,m_t}\mu_t) \right\rangle - \int_{\mathcal{X}\times\mathcal{E}_{\mathcal{X}}} \tau(D\alpha(x,\Delta))\bar{\lambda}_{x,x+d\Delta}(\mu_t,m_t)\mu_t(dx) \Bigg) + \sup_{g\in B(\mathcal{Y})} \int_{\mathcal{Y}} \Bigg(-L_{\mu_t} g(y) - \int_{\mathcal{E}_{\mathcal{Y}}} \tau(Dg(y,\Delta))\gamma_{y,y+d\Delta}(\mu_t) \Bigg) m_t(dy) \Bigg\}\, dt, \tag{3.8}$$

where $m_t \in \mathcal{M}_1(\mathcal{Y})$ is such that θ, when viewed as a measure on $[0,T]\times\mathcal{Y}$, admits the representation $\theta(dydt) = m_t(dy)dt$ for almost all $t\in[0,T]$. Moreover, there exist functions $h_{\mathcal{X}} \in \mathcal{H}_{\mathcal{X}}(\mu,\theta)$ and $h_{\mathcal{Y}} \in \mathcal{H}_{\mathcal{Y}}(\mu,\theta)$ that satisfy

$$\int_{[0,T]\times\mathcal{X}\times\mathcal{E}_{\mathcal{X}}} h_{\mathcal{X}} D\alpha \bar{\lambda}_{x,x+d\Delta}(\mu_t,m_t)\mu_t(dx)dt = \int_{[0,T]} \left\langle \alpha_t, (\dot{\mu}_t - \bar{\Lambda}^*_{\mu_t,m_t}\mu_t) \right\rangle dt,\ \forall \alpha \in B([0,T]\times\mathcal{X}), \tag{3.9}$$

and

$$\int_{[0,T]\times\mathcal{Y}\times\mathcal{E}_{\mathcal{Y}}} h_{\mathcal{Y}} Dg \gamma_{y,y+d\Delta}(\mu_t)m_t(dy)dt = -\int_{[0,T]\times\mathcal{Y}\times\mathcal{E}_{\mathcal{Y}}} Dg\gamma_{y,y+d\Delta}(\mu_t)m_t(dy)dt,\ \forall g \in B([0,T]\times\mathcal{Y}), \tag{3.10}$$

respectively, $h_{\mathcal{X}} \in L^{\tau^*}([0,T]\times\mathcal{X}\times\mathcal{E}_{\mathcal{X}}, \bar{\lambda}_{x,x+d\Delta}\mu_t(dx)dt)$ and $h_{\mathcal{Y}} \in L^{\tau^*}([0,T]\times\mathcal{Y}\times\mathcal{E}_{\mathcal{Y}}, \gamma_{y,y+d\Delta}(\mu_t)m_t(dy)dt)$, and $I^*(\mu,\theta)$ admits the representation

$$I^*(\mu,\theta) = \int_{[0,T]\times\mathcal{X}\times\mathcal{E}_{\mathcal{X}}} \tau^*(h_{\mathcal{X}})\bar{\lambda}_{x,x+d\Delta}(\mu_t,m_t)\mu_t(dx)dt + \int_{[0,T]\times\mathcal{Y}\times\mathcal{E}_{\mathcal{Y}}} \tau^*(h_{\mathcal{Y}})\gamma_{y,y+d\Delta}(\mu_t)m_t(dy)dt. \tag{3.11}$$

Furthermore, if $\inf_{t\in[0,T]}\min_{x\in\mathcal{X}}\mu_t(x) > 0$ and $\inf_{t\in[0,T]}\min_{y\in\mathcal{Y}} m_t(y) > 0$, the suprema in (3.8) over α and g are attained by $\hat{\alpha}_t \in \mathbb{R}^{\mathcal{X}}$ and $\hat{g}_t \in B(\mathcal{Y})$ that satisfy

$$\dot{\mu}_t(x) - (\bar{\Lambda}^*_{\mu_t,m_t}\mu_t)(x)$$

$$+\mu_t(x)\sum_{\substack{x'\in\mathcal{X}:\\(x,x')\in\mathcal{E}_\mathcal{X}}}(\exp\{\hat{\alpha}_t(x')-\hat{\alpha}_t(x)\}-1)\bar{\lambda}_{x,x'}(\mu_t,m_t)$$
$$-\sum_{\substack{x_0\in\mathcal{X}:\\(x_0,x)\in\mathcal{E}_\mathcal{X}}}\mu_t(x_0)(\exp\{\hat{\alpha}_t(x)-\hat{\alpha}_t(x_0)\}-1)\bar{\lambda}_{x_0,x}(\mu_t,m_t)=0,\ \forall x\in\mathcal{X}, \tag{3.12}$$

and

$$m_t(y)\sum_{\substack{y'\in\mathcal{Y}:\\(y,y')\in\mathcal{E}_\mathcal{Y}}}\exp\{\hat{g}_t(y')-\hat{g}_t(y)\}\gamma_{y,y'}(\mu_t)$$
$$-\sum_{\substack{y_0\in\mathcal{Y}:\\(y_0,y)\in\mathcal{E}_\mathcal{Y}}}m_t(y_0)\exp\{\hat{g}_t(y)-\hat{g}_t(y_0)\}\gamma_{y_0,y}(\mu_t)=0,\ \forall y\in\mathcal{Y}, \tag{3.13}$$

for almost all $t\in[0,T]$, respectively.

Proof. For the first part of the theorem, we shall make use of Lemma 3.2. Note that, by Lemma 3.1, we have that the mapping $[0,T]\ni t\mapsto\mu_t\in\mathcal{M}_1(\mathcal{X})$ is absolutely continuous and θ admits the representation $\theta(dydt)=m_t(dy)dt$ where $m_t\in\mathcal{M}_1(\mathcal{Y})$ for almost all $t\in[0,T]$. Therefore, for each $t\geq 0$, $U_t^{\alpha,g}$ in (3.5) can be written as

$$\begin{aligned}U_t^{\alpha,g}(\mu,\theta)=&\int_0^t\langle\alpha_s(\mu_s),\dot{\mu}_s\rangle ds-\int_0^t\langle\alpha_s(\mu_s),\bar{\Lambda}^*_{\mu_s,m_s}\mu_s\rangle ds\\&-\int_0^t\int_{\mathcal{X}\times\mathcal{E}_\mathcal{X}}\tau(D\alpha_s(\mu_s)(x,\Delta))\bar{\lambda}_{x,x+d\Delta}(\mu_s,m_s)\mu_s(dx)ds\\&-\int_0^t\int_\mathcal{Y}\Big(L_{\mu_s}g_s(\mu_s,\cdot)(y)\\&\qquad+\int_{\mathcal{E}_\mathcal{Y}}\tau(Dg_s(\mu_s,y,\Delta))\gamma_{y,y+d\Delta}(\mu_s)\Big)m_s(dy)ds,\end{aligned}$$

where α and g be satisfy the requirements in the definition of $U_t^{\alpha,g}$ in (3.5). Thus,

$$\begin{aligned}I^*(\mu,\theta)=&\sup_\alpha\int_{[0,T]}\Big(\langle\alpha_t(\mu_t),\dot{\mu}_t\rangle-\langle\alpha_t(\mu_t),\bar{\Lambda}^*_{\mu_t,m_t}\mu_t\rangle\\&\qquad-\int_{\mathcal{X}\times\mathcal{E}_\mathcal{X}}\tau(D\alpha_t(\mu_t)(x,\Delta))\bar{\lambda}_{x,x+d\Delta}(\mu_t,m_t)\mu_t(dx)\Big)dt\\&+\sup_g\int_{[0,T]}\int_\mathcal{Y}\Big(-L_{\mu_t}g_t(\mu_t,\cdot)(y)\\&\qquad-\int_{\mathcal{E}_\mathcal{Y}}\tau(Dg_t(\mu_t,y,\Delta))\gamma_{y,y+d\Delta}(\mu_t)\Big)m_t(dy)dt\end{aligned}$$

where the supremum is taken over all functions α and g that satisfy the conditions in the definition of $U_t^{\alpha,g}$ in (3.5). Note that, since μ is kept fixed, an approximation argument using mollifiers implies that the above supremum over α can be replaced by supremum over $\mathbb{R}^{\mathcal{X}}$-valued bounded measurable functions on $[0,T]$. Once again, since μ is fixed, we can replace the supremum over $g \in C^{1,1}([0,T], \mathcal{M}_1(\mathcal{X}) \times \mathcal{Y})$ with the supremum over bounded measurable functions on $[0,T] \times \mathcal{Y}$. Therefore,

$$
\begin{aligned}
I^*(\mu,\theta) = \sup_{\alpha} \int_{[0,T]} \Big(& \langle \alpha_t(\mu_t), \dot{\mu}_t \rangle - \langle \alpha_t(\mu_t), \bar{\Lambda}^*_{\mu_t,m_t}\mu_t \rangle \\
& - \int_{\mathcal{X}\times\mathcal{E}_{\mathcal{X}}} \tau(D\alpha_t(\mu_t)(x,\Delta))\bar{\lambda}_{x,x+d\Delta}(\mu_t,m_t)\mu_t(dx) \Big) dt \\
+ \sup_{g} \int_{[0,T]} \int_{\mathcal{Y}} \Big(& - L_{\mu_t} g_t(\mu_t,\cdot)(y) \\
& - \int_{\mathcal{E}_{\mathcal{Y}}} \tau(Dg_t(\mu_t,y,\Delta))\gamma_{y,y+d\Delta}(\mu_t) \Big) m_t(dy)dt
\end{aligned}
$$

where the supremum is taken over bounded measurable functions $\alpha : [0,T] \to \mathbb{R}^{\mathcal{X}}$ and $g : [0,T] \times \mathcal{Y} \to \mathbb{R}$. We can now apply Lemma 3.2 to conclude that $I^*(\mu,\theta)$ is given by (3.8).

We obtain the existence of functions $h_{\mathcal{X}} \in \mathcal{H}_{\mathcal{X}}(\mu,\theta)$ and $h_{\mathcal{Y}} \in \mathcal{H}_{\mathcal{Y}}(\mu,\theta)$ that satisfy the conditions (3.9) and (3.10) and the non-variational representation of I^* in (3.11) by carrying out the convex analytic programme of Léonard [56, Sections 5-6] to the bounded linear functionals

$$
\alpha \mapsto \int_{[0,T]} \big\langle \alpha_t, (\dot{\mu}_t - \bar{\Lambda}^*_{\mu_t,m_t}\mu_t) \big\rangle \, dt
$$

and

$$
g \mapsto \int_{[0,T]\times\mathcal{Y}\times\mathcal{E}_{\mathcal{Y}}} \big(g_t(y+\Delta) - g_t(y)\big)\gamma_{y,y+d\Delta}(\mu_t)m_t(dy)dt
$$

on the closure of $\{D\alpha, \alpha \in B([0,T]\times\mathcal{X})\}$ and $\{Dg, g \in B([0,T]\times\mathcal{Y})\}$ in the Orlicz spaces $L^\tau([0,T]\times\mathcal{X}\times\mathcal{E}_{\mathcal{X}}, \bar{\lambda}_{x,x+d\Delta}(\mu_t,m_t)\mu_t(dx)dt)$ and $L^\tau([0,T]\times\mathcal{Y}\times\mathcal{E}_{\mathcal{Y}}, \gamma_{y,y+d\Delta}(\mu_t)m_t(dy)dt)$ respectively; the proof follows verbatim from Léonard [56] to our case, and we omit the details here.

Finally, to show the existence of supremisers $\hat{\alpha}_t$ and $\hat{g}_t$ in (3.8) and the conditions (3.12) and (3.13) in the case when $\inf_{t\in[0,T]} \min_{x\in\mathcal{X}} \mu_t(x) > 0$ and $\inf_{t\in[0,T]} \min_{y\in\mathcal{Y}} m_t(y) > 0$, note that, for each $t \in [0,T]$ for which $\dot{\mu}_t$ exists, the mappings

$$\alpha_t \mapsto \langle \alpha_t, (\dot{\mu}_t - \bar{\Lambda}^*_{\mu_t,m_t}\mu_t) \rangle - \int_{\mathcal{X}\times\mathcal{E}_\mathcal{X}} \tau(D\alpha_t(x,\Delta))\bar{\lambda}_{x,x+d\Delta}(\mu_t,m_t)\mu_t(dx) \tag{3.14}$$

and, viewing g_t as an element of $\mathbb{R}^\mathcal{Y}$,

$$g_t \mapsto -\int_\mathcal{Y} \Big(L_{\mu_t} g_t(y) + \int_{\mathcal{E}_\mathcal{Y}} \tau(Dg_t(y,\Delta))\gamma_{y,y+d\Delta}(\mu_t) \Big) m_t(dy) \tag{3.15}$$

are concave on $\mathbb{R}^\mathcal{X}$ and $\mathbb{R}^\mathcal{Y}$ respectively. Therefore, there is an $\hat{\alpha}_t$ and a $\hat{g}_t$ that attain the suprema in (3.8); the conditions in (3.12) and (3.13) on $\hat{\alpha}_t$ and $\hat{g}_t$ easily follow by writing down the first order conditions for optimality of the mappings in (3.14) and (3.15) respectively. $\square$

3.5 Characterisation of the subsequential rate function for sufficiently regular elements

Let $\tilde{I} : D([0,T],\mathcal{M}_1(\mathcal{X})) \times D_\uparrow([0,T],\mathcal{M}(\mathcal{Y})) \to [0,+\infty]$ be a subsequential rate function for the family $\{(\mu^N,\theta^N), N \geq 1\}$, i.e., for some sequence $\{N_k, k \geq 1\}$ of $\mathbb{N}$, $\{(\mu^{N_k},\theta^{N_k}), k \geq 1\}$ satisfies the large deviation principle with rate function $\tilde{I}$. In addition suppose that, for some $\nu \in \mathcal{M}_1(\mathcal{X})$, $\tilde{I}(\mu,\theta) = +\infty$ unless $\mu_0 = \nu$. In this section, we characterise $\tilde{I}$ for sufficiently regular elements in $D([0,T],\mathcal{M}_1(\mathcal{X})) \times D_\uparrow([0,T],\mathcal{M}(\mathcal{Y}))$, i.e., we show that $\tilde{I}(\hat{\mu},\hat{\theta}) = I^*(\hat{\mu},\hat{\theta})$ for all elements $(\hat{\mu},\hat{\theta}) \in D([0,T],\mathcal{M}_1(\mathcal{X})) \times D_\uparrow([0,T],\mathcal{M}(\mathcal{Y}))$ that satisfy certain regularity properties, where I^* is given by (3.8) (see Theorem 3.8).

3.5.1 An extension of Theorem 3.5

We first extend the conclusion of Theorem 3.5 to a larger class of functions α and g. Let $\Gamma \subset D([0,T],\mathcal{M}_1(\mathcal{X})) \times D_\uparrow([0,T],\mathcal{M}(\mathcal{Y}))$ denote the set of points (μ,θ) such that the mapping $[0,T] \ni t \mapsto \mu_t \in \mathcal{M}_1(\mathcal{X})$ is absolutely continuous, and θ, when viewed as a measure on $[0,T]\times\mathcal{Y}$, admits the representation $\theta(dydt) = m_t(dy)dt$ where $m_t \in \mathcal{M}_1(\mathcal{Y})$ for almost all $t \in [0,T]$. In particular, $(\mu,\theta) \in \Gamma$ implies that the mapping $t \mapsto \mu_t$ is differentiable for almost all $t \in [0,T]$. Given bounded measurable functions $\alpha : [0,T]\times\mathcal{M}_1(\mathcal{X}) \to \mathbb{R}^\mathcal{X}$ and $g : [0,T]\times\mathcal{M}_1(\mathcal{X})\times\mathcal{Y} \to \mathbb{R}$ such that for all $t \in [0,T]$ and $y \in \mathcal{Y}$ both $\alpha(t,\cdot)$ and $g(t,\cdot,y)$ are continuous on $\mathcal{M}_1(\mathcal{X})$, we define, with a slight abuse of notation, for $(\mu,\theta)\in\Gamma$ and $t\in[0,T]$,

$$U_t^{\alpha,g}(\mu,\theta) := \int_{[0,t]} \Big\{ \langle \alpha_s(\mu_s), \dot{\mu}_s - \bar{\Lambda}^*_{\mu_s,m_s}\mu_s \rangle - \int_{\mathcal{X}\times\mathcal{E}_\mathcal{X}} \tau(D\alpha_s(\mu_s)(x,\Delta))\bar{\lambda}_{x,x+d\Delta}(\mu_s,m_s)\mu_s(dx) - \int_\mathcal{Y}\Big(L_{\mu_s} g_s(\mu_s,\cdot)(y) + \int_{\mathcal{E}_\mathcal{Y}} \tau(Dg_s(\mu_s,y,\Delta))\gamma_{y,y+d\Delta}(\mu_s)\Big) m_s(dy)\Big\} ds. \tag{3.16}$$

Note that the boundedness of α and g in the above definition implies that $D\alpha\in L^\tau([0,T]\times\mathcal{X}\times\mathcal{E}_\mathcal{X},\bar{\lambda}_{x,x+d\Delta}(\mu_t,m_t)\mu_t(dx)dt)$, and $Dg\in L^\tau([0,T]\times\mathcal{E}_\mathcal{Y}\times\mathcal{Y},\gamma_{y,y+d\Delta}(\mu_t)m_t(dy)dt)$.

Let $\tilde{I}: D([0,T],\mathcal{M}_1(\mathcal{X}))\times D_\uparrow([0,T],\mathcal{M}(\mathcal{Y}))\to[0,+\infty]$ be a subsequential rate function for the family $\{(\mu^N,\theta^N), N\geq 1\}$. Note that, by Theorem 3.5 and the definition of I^* in (3.7), we have that $\tilde{I}(\mu,\theta)\geq I^*(\mu,\theta)$ for all $(\mu,\theta)\in D([0,T],\mathcal{M}_1(\mathcal{X}))\times D_\uparrow([0,T],\mathcal{M}(\mathcal{Y}))$. Given $\delta>0$, define

$$K_\delta = \{(\mu,\theta) : \tilde{I}(\mu,\theta)\leq\delta\};$$

since $\tilde{I}$ has compact level sets, K_δ is compact in $D([0,T],\mathcal{M}_1(\mathcal{X}))\times D_\uparrow([0,T],\mathcal{M}(\mathcal{Y}))$. By Lemma 3.1 and the fact that $\tilde{I}\geq I^*$, we have that $K_\delta\subset\Gamma$. We now prove the following extension to Theorem 3.5.

Theorem 3.7. *Let $\tilde{I}: D([0,T],\mathcal{M}_1(\mathcal{X}))\times D_\uparrow([0,T],\mathcal{M}(\mathcal{Y}))\to[0,+\infty]$ be a subsequential rate function. Let $\alpha:[0,T]\times\mathcal{M}_1(\mathcal{X})\to\mathbb{R}^\mathcal{X}$, $g:[0,T]\times\mathcal{M}_1(\mathcal{X})\times\mathcal{Y}\to\mathbb{R}$ be bounded and measurable functions such that both α and g are continuous on $\mathcal{M}_1(\mathcal{X})$. Then,*

$$\sup_{(\mu,\theta)\in\Gamma}(U_T^{\alpha,g}(\mu,\theta)-\tilde{I}(\mu,\theta))=0.$$

Moreover, there exists some $\delta>0$ (depending on α and g) such that

$$\sup_{(\mu,\theta)\in K_\delta}(U_T^{\alpha,g}(\mu,\theta)-\tilde{I}(\mu,\theta))=0, \tag{3.17}$$

and the above supremum is attained.

Proof. We first define certain approximations of functions α and g that meet the requirements of Theorem 3.5 and prove certain convergence properties of these approximations. We then use the conclusion of Theorem 3.5 for these approximations and pass to the limit to obtain (3.17). Our proof is inspired by ideas from Puhalskii [71, Lemma 7.2 and Theorem 7.1], with necessary modifications to our mean-field with jumps setting.

Since α is a Carathéodory function, using the Scorza-Dragoni theorem, for each $i \geq 1$, there exists a compact set $F_i \subset [0,T]$ and a measurable function $\bar{\alpha}_i : [0,T] \times \mathcal{M}_1(\mathcal{X}) \to \mathbb{R}^{\mathcal{X}}$ such that $\bar{\alpha}_i = \alpha$ on $F_i \times \mathcal{M}_1(\mathcal{X})$, $\bar{\alpha}_i$ is continuous on $F_i \times \mathcal{M}_1(\mathcal{X})$, and $\text{Leb}([0,T] \setminus F_i) \leq 1/i$ (see, for example, Ekeland and Temam [33, page 235]). Since $[0,T] \setminus F_i$ is open in $[0,T]$, we can write it as a countable union of disjoint open intervals, and hence we can extend $\bar{\alpha}_i$ to a continuous function on $[0,T] \times \mathcal{M}_1(\mathcal{X})$ by a linear interpolation between the two endpoints of the above open intervals; we again denote this function by $\bar{\alpha}_i$. Put $\alpha_i(t,\mu_t) = \bar{\alpha}_i(\frac{\lfloor tn(i) \rfloor}{n(i)}, \mu_{\frac{\lfloor tn(i) \rfloor}{n(i)}})$, where $n(i) \to \infty$ as $i \to \infty$. By continuity of τ, boundedness of α and α_i, boundedness of transition rates of the particles (which is a consequence of assumption (C2)), we have that, for each $\delta > 0$,

$$\begin{aligned}
&\sup_{(\mu,\theta)\in K_\delta} \Bigg| \int_{[0,T]\times\mathcal{X}\times\mathcal{E}_\mathcal{X}} \tau(D\alpha_i(t,\mu_t)(x,\Delta))\bar{\lambda}_{x,x+d\Delta}(\mu_t,m_t)\mu_t(dx)dt - \\
&\qquad \int_{[0,T]\times\mathcal{X}\times\mathcal{E}_\mathcal{X}} \tau(D\alpha(t,\mu_t)(x,\Delta))\bar{\lambda}_{x,x+d\Delta}(\mu_t,m_t)\mu_t(dx)dt \Bigg| \\
&= \sup_{(\mu,\theta)\in K_\delta} \Bigg| \int_{F_i^c\times\mathcal{X}\times\mathcal{E}_\mathcal{X}} \tau(D\alpha_i(t,\mu_t)(x,\Delta))\bar{\lambda}_{x,x+d\Delta}(\mu_t,m_t)\mu_t(dx)dt - \\
&\qquad \int_{F_i^c\times\mathcal{X}\times\mathcal{E}_\mathcal{X}} \tau(D\alpha(t,\mu_t)(x,\Delta))\bar{\lambda}_{x,x+d\Delta}(\mu_t,m_t)\mu_t(dx)dt \Bigg| \\
&\leq \text{Leb}(F_i^c) \times c_\alpha \to 0
\end{aligned} \tag{3.18}$$

as $i \to \infty$, where $c_\alpha > 0$ is a constant depending on α. Furthermore, given $\delta > 0$ and $(\mu,\theta) \in K_\delta$, by Lemma 3.1, the mapping $[0,T] \ni t \mapsto \mu_t \in \mathcal{M}_1(\mathcal{X})$ is absolutely continuous. Hence, noting that μ is kept fixed, by (3.9) in Theorem 3.6, there exists $h_\mathcal{X} \in \mathcal{H}(\mu,\theta)$ such that

$$\begin{aligned}
&\int_{[0,T]} \left\langle \alpha(t,\mu_t), (\dot{\mu}_t - \bar{\Lambda}^*_{\mu_t,m_t}\mu_t) \right\rangle dt \\
&\qquad = \int_{[0,T]\times\mathcal{X}\times\mathcal{E}_\mathcal{X}} h_\mathcal{X} D\alpha \bar{\lambda}_{x,x+d\Delta}(\mu_t,m_t)\mu_t(dx)dt,
\end{aligned}$$

and

$$\begin{aligned}
&\int_{[0,T]} \left\langle \alpha_i(t,\mu_t), (\dot{\mu}_t - \bar{\Lambda}^*_{\mu_t,m_t}\mu_t) \right\rangle dt \\
&\qquad = \int_{[0,T]\times\mathcal{X}\times\mathcal{E}_\mathcal{X}} h_\mathcal{X} D\alpha_i \bar{\lambda}_{x,x+d\Delta}(\mu_t,m_t)\mu_t(dx)dt.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\Big|\int_{[0,T]} &\langle \alpha_i(t,\mu_t) - \alpha(t,\mu_t), \dot{\mu}_t - \bar{\Lambda}^*_{\mu_t,m_t}\mu_t\rangle dt\Big| \\
&= \Big|\int_{[0,T]\times\mathcal{X}\times\mathcal{E}_{\mathcal{X}}} h_{\mathcal{X}}(D\alpha_i - D\alpha)\bar{\lambda}_{x,x+d\Delta}(\mu_t,m_t)\mu_t(dx)dt\Big| \\
&\le \int_{[0,T]\times\mathcal{X}\times\mathcal{E}_{\mathcal{X}}} |h_{\mathcal{X}}(D\alpha_i - D\alpha)|\bar{\lambda}_{x,x+d\Delta}(\mu_t,m_t)\mu_t(dx)dt \\
&\le 2\|h_{\mathcal{X}}\|_{L^{\tau^*}([0,T]\times\mathcal{X}\times\mathcal{E}_{\mathcal{X}},\bar{\lambda}_{x,x+d\Delta}(\mu_t,m_t)\mu_t(dx)dt)} \\
&\qquad \times \|D\alpha_i - D\alpha\|_{L^{\tau}([0,T]\times\mathcal{X}\times\mathcal{E}_{\mathcal{X}},\bar{\lambda}_{x,x+d\Delta}(\mu_t,m_t)\mu_t(dx)dt)} \\
&\le 2\max\{1,\delta+T\} \\
&\qquad \times \|D\alpha_i - D\alpha\|_{L^{\tau}([0,T]\times\mathcal{X}\times\mathcal{E}_{\mathcal{X}},\bar{\lambda}_{x,x+d\Delta}(\mu_t,m_t)\mu_t(dx)dt)},
\end{aligned}$$

where the second inequality follows from Hölder's inequality in Orlicz spaces and the third inequality follows from the non-variational representation of the candidate rate function in I^* in (3.11), which gives that $\|h_{\mathcal{X}}\|_{L^{\tau^*}([0,T]\times\mathcal{X}\times\mathcal{E}_{\mathcal{X}},\bar{\lambda}_{x,x+d\Delta}(\mu_t,m_t)\mu_t(dx)dt)} \le \max\{1, I^*(\mu,\theta)+T\}$, along with the fact that $(\mu,\theta)\in K_\delta$ and $I^*(\mu,\theta)\le \tilde{I}(\mu,\theta)$. Hence,

$$\sup_{(\mu,\theta)\in K_\delta}\Big|\int_{[0,T]}\langle \alpha_i(t,\mu_t) - \alpha(t,\mu_t), \dot{\mu}_t - \bar{\Lambda}^*_{\mu_t,m_t}\mu_t\rangle dt\Big| \to 0 \tag{3.19}$$

as $i\to\infty$. Similarly, by standard arguments using mollifiers and the Scorza-Dragoni theorem, we can show that there exist functions g_i on $[0,T]\times\mathcal{M}_1(\mathcal{X})\times\mathcal{Y}$ such that $g_i(\cdot,\cdot,y)\in C^\infty([0,T]\times\mathcal{M}_1(\mathcal{X}))$ for all $y\in\mathcal{Y}$ and $\text{Leb}\{t\in[0,T] : g_i(t,\cdot,\cdot)\neq g(t,\cdot,\cdot)\}\le 1/i$ for each $i\ge 1$. Therefore, using boundedness of the functions g, $g_i, i\ge 1$, and boundedness of the transition rates of the fast process (which is a consequence of assumption (D2)), we see that

$$\begin{aligned}
\sup_{(\mu,\theta)\in K_\delta}\Big|\int_{[0,T]\times\mathcal{Y}} &\Big(L_{\mu_t}g_i(t,\mu_t,\cdot)(y) \\
&+ \int_{\mathcal{E}_{\mathcal{Y}}}\tau(Dg_i(t,\mu_t,y,\Delta))\gamma_{y,y+d\Delta}(\mu_t)\Big)m_t(dy)dt \\
&- \int_{[0,T]\times\mathcal{Y}}\Big(L_{\mu_t}g(t,\mu_t,\cdot)(y) \\
&+ \int_{\mathcal{E}_{\mathcal{Y}}}\tau(Dg(t,\mu_t,y,\Delta))\gamma_{y,y+d\Delta}(\mu_t)\Big)m_t(dy)dt\Big| \to 0
\end{aligned} \tag{3.20}$$

as $i\to\infty$. Since α_i and g_i, $i\ge 1$, satisfy the conditions on α and g respectively in the definitions of U in (3.5) and V in (3.3), Theorem 3.5 implies that

$$\sup_{(\mu,\theta)\in D([0,T],\mathcal{M}_1(\mathcal{X}))\times D_{\uparrow}([0,T],\mathcal{M}(\mathcal{Y}))} (U_T^{\alpha_i,g_i}(\mu,\theta) - \tilde{I}(\mu,\theta)) = 0.$$

By Lemma 3.1 and the fact that $\tilde{I}(\mu,\theta) \geq I^*(\mu,\theta)$, we see that $\tilde{I}(\mu,\theta) = +\infty$ whenever $(\mu,\theta) \notin \Gamma$, and hence we immediately get

$$\sup_{(\mu,\theta)\in\Gamma} (U_T^{\alpha_i,g_i}(\mu,\theta) - \tilde{I}(\mu,\theta)) = 0. \tag{3.21}$$

Let us now show that

$$\sup_{(\mu,\theta)\in K_\delta} (U_T^{\alpha_i,g_i}(\mu,\theta) - \tilde{I}(\mu,\theta)) = 0 \tag{3.22}$$

holds for a suitable $\delta > 0$ and all $i \geq 1$. Note that, using the boundedness of the functions α, g, α_i and g_i, $i \geq 1$, and the boundedness of the transition rates (as a consequence of assumptions (C2) and (D2)), we have

$$\begin{aligned} U_T^{2\alpha_i,2g_i}(\mu,\theta) = \int_{[0,T]} \Bigg\{ & 2\langle \alpha_i(t,\mu_t), \dot{\mu}_t - \bar{\Lambda}^*_{\mu_t,m_t}\mu_t \rangle \\ & - \int_{\mathcal{X}\times\mathcal{E}_{\mathcal{X}}} \tau(2D\alpha_i(t,\mu_t)(x,\Delta))\bar{\lambda}_{x,x+d\Delta}(\mu_t,m_t)\mu_t(dx) \\ & - \int_{\mathcal{Y}} \Bigg(2L_{\mu_t} g_t(\mu_t,\cdot)(y) \\ & + \int_{\mathcal{E}_{\mathcal{Y}}} \tau(2Dg_i(t,\mu_t,y,\Delta))\gamma_{y,y+d\Delta}(\mu_t) \Bigg) m_t(dy) \Bigg\} dt \\ \geq {} & 2U_T^{\alpha_i,g_i}(\mu,\theta) - 2Tc_{\alpha,g} \end{aligned}$$

for all $i \geq 1$, where $c_{\alpha,g} > 0$ is a constant depending on α and g. Therefore, for a fixed $M > 0$, we have

$$\begin{aligned} & \sup_{(\mu,\theta):U_T^{\alpha_i,g_i}(\mu,\theta)\geq M} (U_T^{\alpha_i,g_i}(\mu,\theta) - \tilde{I}(\mu,\theta)) \\ & \qquad \leq \sup_{(\mu,\theta):U_T^{\alpha_i,g_i}(\mu,\theta)\geq M} (2U_T^{\alpha_i,g_i}(\mu,\theta) - \tilde{I}(\mu,\theta)) - M \\ & \qquad \leq \sup_{(\mu,\theta):U_T^{\alpha_i,g_i}(\mu,\theta)\geq M} (U_T^{2\alpha_i,2g_i}(\mu,\theta) - \tilde{I}(\mu,\theta)) + 2Tc_{\alpha,g} - M \\ & \qquad \leq 2Tc_{\alpha,g} - M. \end{aligned}$$

Therefore the above implies that,

$$\begin{aligned}
\sup_{(\mu,\theta)\in\Gamma}&(U_T^{\alpha_i,g_i}(\mu,\theta)-\tilde{I}(\mu,\theta))\\
&\le \sup_{(\mu,\theta)\in K_\delta}(U_T^{\alpha_i,g_i}(\mu,\theta)-\tilde{I}(\mu,\theta))\\
&\quad\vee \sup_{(\mu,\theta):U_T^{\alpha_i,g_i}(\mu,\theta)\ge M}(U_T^{\alpha_i,g_i}(\mu,\theta)-\tilde{I}(\mu,\theta))\\
&\quad\vee (M-\delta)\\
&\le \sup_{(\mu,\theta)\in K_\delta}(U_T^{\alpha_i,g_i}(\mu,\theta)-\tilde{I}(\mu,\theta))\vee(2Tc_{\alpha,g}-M)\vee(M-\delta).
\end{aligned}$$

Hence, choosing $M=1+2Tc_{\alpha,g}$ and $\delta=M+1$, the above and (3.21) imply (3.22). Letting $i\to\infty$, using convergences (3.18)-(3.19) for the slow process, and (3.20) for the fast process, (3.22) becomes

$$\sup_{(\mu,\theta)\in K_\delta}(U_T^{\alpha,g}(\mu,\theta)-\tilde{I}(\mu,\theta))=0. \tag{3.23}$$

Since the functions $U_T^{\alpha_i,g_i}$ (defined in (3.5)), $i\ge 1$, are continuous on Γ and since for all $\delta'>0$

$$\lim_{i\to\infty}\sup_{(\mu,\theta)\in K_{\delta'}}|U_T^{\alpha_i,g_i}(\mu,\theta)-U_T^{\alpha,g}(\mu,\theta)|\to 0$$

as $i\to\infty$, it follows that, for all $\delta'>0$, $U_T^{\alpha,g}$ (defined in (3.16)) is continuous on $K_{\delta'}$. Hence, using the compactness of the level sets of $\tilde{I}$, we see that the supremum in (3.23) is attained. This completes the proof of the theorem. □

3.5.2 Characterisation of $\tilde{I}$ for regular elements

We now prove the main result of this section, namely $\tilde{I}(\mu,\theta)=I^*(\mu,\theta)$ for all $(\mu,\theta)\in D([0,T],\mathcal{M}_1(\mathcal{X}))\times D_\uparrow([0,T],\mathcal{M}(\mathcal{Y}))$ that satisfy certain regularity properties.

Theorem 3.8. *Let $\nu\in\mathcal{M}_1(\mathcal{X})$ and let $\tilde{I}:D([0,T],\mathcal{M}_1(\mathcal{X}))\times D_\uparrow([0,T],\mathcal{M}(\mathcal{Y}))\to[0,+\infty]$ be a subsequential rate function such that $\tilde{I}(\mu,\theta)=+\infty$ unless $\mu_0=\nu$. Suppose that $(\hat{\mu},\hat{\theta})\in D([0,T],\mathcal{M}_1(\mathcal{X}))\times D_\uparrow([0,T],\mathcal{M}(\mathcal{Y}))$ is such that*

- $\inf_{t\in[0,T]}\min_{x\in\mathcal{X}}\hat{\mu}_t(x)>0$,
- *the mapping $[0,T]\ni t\mapsto\hat{\mu}_t\in\mathcal{M}_1(\mathcal{X})$ is Lipschitz continuous,*

- $\hat{\theta}$, *when viewed as a measure on* $[0,T]\times\mathcal{Y}$, *admits the representation* $\hat{\theta}(dydt)=\hat{m}_t(dy)dt$ *for some* $\hat{m}_t\in\mathcal{M}_1(\mathcal{Y})$ *for almost all* $t\in[0,T]$, *and* $\inf_{t\in[0,T]}\min_{y\in\mathcal{Y}}\hat{m}_t(y)>0$.

Then $\tilde{I}(\hat{\mu},\hat{\theta})=I^*(\hat{\mu},\hat{\theta})$.

Proof. Let $\delta=\inf_{t>0}\min_{x\in\mathcal{X}}\hat{\mu}_t(x)$. For each $t\in[0,T]$, consider the parametrised optimisation problems

$$\sup_{\alpha_t\in\mathbb{R}^{\mathcal{X}}}\left\{\langle\alpha_t,\dot{\hat{\mu}}_t-\bar{\Lambda}^*_{u,\hat{m}_t}u\rangle-\int_{\mathcal{X}\times\mathcal{E}_{\mathcal{X}}}\tau(D\alpha_t(x,\Delta))\bar{\lambda}_{x,x+d\Delta}(u,\hat{m}_t)u(dx)\right\}, \tag{3.24}$$

$u\in\mathcal{M}_1(\mathcal{X})$ is such that $u(x)\geq\delta/2$ for all $x\in\mathcal{X}$, and

$$\sup_{g_t\in B(\mathcal{Y})}\left\{-\int_{\mathcal{Y}}\left(L_u g_t(\cdot)(y)+\int_{\mathcal{E}_{\mathcal{Y}}}\tau(Dg_t(y,\Delta))\gamma_{y,y+d\Delta}(u)\right)\hat{m}_t(dy)\right\}, \tag{3.25}$$

$u\in\mathcal{M}_1(\mathcal{X})$. Note that the mappings

$$\alpha_t\mapsto\langle\alpha_t,\dot{\hat{\mu}}_t-\bar{\Lambda}^*_{u,\hat{m}_t}u\rangle-\int_{\mathcal{X}\times\mathcal{E}_{\mathcal{X}}}\tau(D\alpha_t(x,\Delta))\bar{\lambda}_{x,x+d\Delta}(u,\hat{m}_t)u(dx), \tag{3.26}$$

where u is such that $u(x)\geq\delta/2$ for all $x\in\mathcal{X}$, and since $\inf_{t\in[0,T]}\min_{y\in\mathcal{Y}}\hat{m}_t(y)>0$, viewing g_t as an element of $\mathbb{R}^{\mathcal{Y}}$,

$$g_t\mapsto-\int_{\mathcal{Y}}\left(L_u g_t(\cdot)(y)+\int_{\mathcal{E}_{\mathcal{Y}}}\tau(Dg_t(y,\Delta))\gamma_{y,y+d\Delta}(u)\right)\hat{m}_t(dy) \tag{3.27}$$

are concave on $\mathbb{R}^{\mathcal{X}}$ and $\mathbb{R}^{\mathcal{Y}}$ respectively. Therefore, we see that there exist an $\hat{\alpha}_t(u)\in\mathbb{R}^{\mathcal{X}}$ and a $\hat{g}_t(u)\in\mathbb{R}^{\mathcal{Y}}$ that solve (3.24) and (3.25) respectively. Guided by (3.12) and (3.13), $\hat{\alpha}_t(u)$ and $\hat{g}_t(u)$ satisfy the first order optimality conditions

$$\begin{aligned}&\dot{\hat{\mu}}_t(x)-(\bar{\Lambda}^*_{u,\hat{m}_t}u)(x)\\&\qquad+u(x)\sum_{\substack{x'\in\mathcal{X}:\\(x,x')\in\mathcal{E}_{\mathcal{X}}}}(\exp\{\hat{\alpha}_t(u)(x')-\hat{\alpha}_t(u)(x)\}-1)\bar{\lambda}_{x,x'}(u,\hat{m}_t)\\&\qquad-\sum_{\substack{x_0\in\mathcal{X}:\\(x_0,x)\in\mathcal{E}_{\mathcal{X}}}}u(x_0)(\exp\{\hat{\alpha}_t(u)(x)-\hat{\alpha}_t(u)(x_0)\}-1)\bar{\lambda}_{x_0,x}(u,\hat{m}_t)=0,\ \forall x\in\mathcal{X},\end{aligned} \tag{3.28}$$

where $t \in [0,T]$ and $u \in \mathcal{M}_1(\mathcal{X})$ is such that $u(x) \geq \delta/2$ for all $x \in \mathcal{X}$, and

$$\hat{m}_t(y) \sum_{\substack{y' \in \mathcal{Y}: \\ (y,y') \in \mathcal{E}_\mathcal{Y}}} \exp\{\hat{g}_t(u,y') - \hat{g}_t(u,y)\}\gamma_{y,y'}(u) - \sum_{\substack{y_0 \in \mathcal{Y}: \\ (y_0,y) \in \mathcal{E}_\mathcal{Y}}} \hat{m}_t(y_0) \exp\{\hat{g}_t(u,y) - \hat{g}_t(u,y_0)\}\gamma_{y_0,y}(u) = 0, \ \forall y \in \mathcal{Y}, \tag{3.29}$$

where $t \in [0,T]$ and $u \in \mathcal{M}_1(\mathcal{X})$, respectively.

We now define bounded measurable functions $\hat{\alpha} : [0,T] \times \mathcal{M}_1(\mathcal{X}) \to \mathbb{R}^\mathcal{X}$ and $\hat{g} : [0,T] \times \mathcal{M}_1(\mathcal{X}) \times \mathcal{Y} \to \mathbb{R}$ that are continuous on $\mathcal{M}_1(\mathcal{X})$ such that $\hat{\alpha}(u)$ (resp. $\hat{g}(u)$) solves the optimisation problem in (3.24) (resp. (3.25)). Note that the objective function in (3.25) is uniquely determined by $\{g(t,y') - g(t,y), (y,y') \in \mathcal{E}_\mathcal{Y}\}$, and by assumption (C1), the objective function in (3.24) is uniquely determined by $\{\alpha_t(x') - \alpha_t(x), (x,x') \in \mathcal{E}_\mathcal{X}\}$. Since $\inf_{t\in[0,T]} \min_{x\in\mathcal{X}} \hat{\mu}_t(x) > 0$, the mapping $t \mapsto \hat{\mu}_t$ is Lipschitz continuous, and the transition rates of the slow process are bounded (which is a consequence of assumption (C2)), we see that we can restrict the supremum over α_t in (3.24) to a single compact and convex subset of $\mathbb{R}^\mathcal{X}$, regardless of $t \in [0,T]$ and $u \in \mathcal{M}_1(\mathcal{X})$ with $u(x) \geq \delta/2$ for all $x \in \mathcal{X}$. Similarly, since $\inf_{t\in[0,T]} \min_{y\in\mathcal{Y}} \hat{m}_t(y) > 0$ and the transition rates of the fast process are bounded (which follows from assumption (D2)), we see that we can restrict the supremum in (3.25) to a single compact and convex subset of $\mathbb{R}^\mathcal{Y}$, regardless of $t \in [0,T]$ and $u \in \mathcal{M}_1(\mathcal{X})$. Also, note that the mappings (3.24) and (3.25), when viewed as

$$\{\alpha_t(x') - \alpha_t(x), (x,x') \in \mathcal{E}_\mathcal{X}\} \mapsto \langle \alpha_t, \dot{\hat{\mu}}_t - \bar{\Lambda}^*_{u,\hat{m}_t} u\rangle - \int_{\mathcal{X}\times\mathcal{E}_\mathcal{X}} \tau(D\alpha_t(x,\Delta))\bar{\lambda}_{x,x+d\Delta}(u,\hat{m}_t)u(dx)$$

and,

$$\{g_t(y') - g_t(y), (y,y') \in \mathcal{E}_\mathcal{Y}\} \mapsto -\int_\mathcal{Y} \left(L_u g_t(\cdot)(y) + \int_{\mathcal{E}_\mathcal{Y}} \tau(Dg_t(y,\Delta))\gamma_{y,y+d\Delta}(u)\right) \hat{m}_t(dy)$$

are strictly concave on $\mathbb{R}^{\mathcal{E}_\mathcal{X}}$ and $\mathbb{R}^{\mathcal{E}_\mathcal{Y}}$ respectively; hence there exists a unique $\{\hat{\alpha}_t(u)(x') - \hat{\alpha}_t(u)(x), (x,x') \in \mathcal{E}_\mathcal{X}\}$ and a unique $\{\hat{g}_t(u,y') - \hat{g}_t(u,y), (y,y') \in \mathcal{E}_\mathcal{Y}\}$ that solve (3.24) and (3.25) respectively. Fixing $\hat{\alpha}_t(u)(x_0) = 0$ for some $x_0 \in \mathcal{X}$, where $t \in [0,T]$ and $u \in \mathcal{M}_1(\mathcal{X})$ with $u(x) \geq \delta/2$ for all $x \in \mathcal{X}$, fixing $g_t(u,y_0) = 0$ for some $y_0 \in \mathcal{Y}$, where $t \in [0,T]$ and $u \in \mathcal{M}_1(\mathcal{X})$, defining $\hat{\alpha}_t(u)(x) = 0\,\forall x \in \mathcal{X}$ whenever $u \in \mathcal{M}_1(\mathcal{X})$ is such that $u(x) < \delta/4$ for some $x \in \mathcal{X}$, and defining $\hat{\alpha}(u)$ whenever u is such that $u(x) \in [\delta/4, \delta/2]$ for some $x \in \mathcal{X}$ using a linear interpolation, we obtain bounded functions $\hat{\alpha} : [0,T] \times \mathcal{M}_1(\mathcal{X}) \to \mathbb{R}^{\mathcal{X}}$ and $\hat{g} : [0,T] \times \mathcal{M}_1(\mathcal{X}) \times \mathcal{Y} \to \mathbb{R}$. By a measurable selection theorem (see, for example, Ekeland and Temam [33, Theorem 1.2, page 236]), it follows that the mappings $[0,T] \times \mathcal{M}_1(\mathcal{X}) \ni (t,u) \mapsto \hat{\alpha}_t(u) \in \mathbb{R}^{\mathcal{X}}$ and $[0,T] \times \mathcal{M}_1(\mathcal{X}) \times \mathcal{Y} \ni (t,u,y) \mapsto \hat{g}_t(u,y) \in \mathbb{R}$ are measurable. By the Berge's maximum theorem (see, for example, Sundaram [82, Theorem 9.17, page 237]) it follows that the functions $\hat{\alpha}$ and $\hat{g}$ are continuous on $\mathcal{M}_1(\mathcal{X})$.

Since $\hat{\alpha}$ and $\hat{g}$ satisfy the assumptions of Theorem 3.7, there exists $(\tilde{\mu}, \tilde{\theta}) \in \Gamma$ that attains the supremum in (3.17) with $\hat{\alpha}$ and $\hat{g}$ in place of α and g, respectively. That is,

$$U_T^{\hat{\alpha},\hat{g}}(\tilde{\mu}, \tilde{\theta}) = \tilde{I}(\tilde{\mu}, \tilde{\theta}).$$

On the other hand, by (3.8) and the above,

$$I^*(\tilde{\mu}, \tilde{\theta}) \geq U_T^{\hat{\alpha},\hat{g}}(\tilde{\mu}, \tilde{\theta}) = \tilde{I}(\tilde{\mu}, \tilde{\theta}),$$

and since $\tilde{I}(\tilde{\mu}, \tilde{\theta}) \geq I^*(\tilde{\mu}, \tilde{\theta})$, we have that

$$U_T^{\hat{\alpha},\hat{g}}(\tilde{\mu}, \tilde{\theta}) = I^*(\tilde{\mu}, \tilde{\theta}) = \tilde{I}(\tilde{\mu}, \tilde{\theta}). \tag{3.30}$$

Note that $\tilde{\mu}_0 = \nu$ since $\tilde{I}(\tilde{\mu}, \tilde{\theta}) < +\infty$. We now proceed to show that $\tilde{m}_t = \hat{m}_t$ for almost all $t \in [0,T]$ and $\tilde{\mu} = \hat{\mu}$. This would establish $\tilde{I}(\hat{\mu}, \hat{\theta}) = I^*(\hat{\mu}, \hat{\theta})$.

By (3.30), we have

$$\begin{aligned} \tilde{m}_t(y) &\sum_{\substack{y' \in \mathcal{Y}: \\ (y,y') \in \mathcal{E}_{\mathcal{Y}}}} \exp\{\hat{g}_t(\tilde{\mu}_t, y') - \hat{g}_t(\tilde{\mu}_t, y)\}\gamma_{y,y'}(\tilde{\mu}_t) \\ &- \sum_{\substack{y_0 \in \mathcal{Y}: \\ (y_0,y) \in \mathcal{E}_{\mathcal{Y}}}} \tilde{m}_t(y_0) \exp\{\hat{g}_t(\tilde{\mu}_t, y) - \hat{g}_t(\tilde{\mu}_t, y_0)\}\gamma_{y_0,y}(\tilde{\mu}_t) = 0, \ \forall y \in \mathcal{Y}, \end{aligned} \tag{3.31}$$

for almost all $t \in [0,T]$. By assumption (D2), the Markov process on $\mathcal{Y}$ with transition rates $\exp\{\hat{g}_t(\tilde{\mu}_t, y') - \hat{g}_t(\tilde{\mu}_t, y)\}\gamma_{y,y'}(\tilde{\mu}_t)$, $(y,y') \in \mathcal{E}_{\mathcal{Y}}$, possesses a unique invariant probability measure; comparing (3.29) with $u = \tilde{\mu}_t$ and (3.31), we get

$$\tilde{m}_t = \hat{m}_t \tag{3.32}$$

for almost all $t \in [0, T]$.

On one hand, by using the first order optimality condition in (3.28) with $u = \hat{\mu}_t$, and the just established fact that $\tilde{m}_t = \hat{m}_t$ for almost all $t \in [0, T]$, we get

$$\begin{aligned} \dot{\hat{\mu}}_t(x) &- (\bar{\Lambda}^*_{\hat{\mu}_t, \tilde{m}_t} \hat{\mu}_t)(x) \\ &+ \hat{\mu}_t(x) \sum_{\substack{x' \in \mathcal{X}: \\ (x,x') \in \mathcal{E}_{\mathcal{X}}}} (\exp\{\hat{\alpha}_t(\hat{\mu}_t)(x') - \hat{\alpha}_t(\hat{\mu}_t)(x)\} - 1) \bar{\lambda}_{x,x'}(\hat{\mu}_t, \tilde{m}_t) \\ &- \sum_{\substack{x_0 \in \mathcal{X}: \\ (x_0,x) \in \mathcal{E}_{\mathcal{X}}}} \hat{\mu}_t(x_0)(\exp\{\hat{\alpha}_t(\hat{\mu}_t)(x) - \hat{\alpha}_t(\hat{\mu}_t)(x_0)\} - 1) \bar{\lambda}_{x_0,x}(\hat{\mu}_t, \tilde{m}_t) = 0, \ \forall x \in \mathcal{X}, \end{aligned} \tag{3.33}$$

for almost all $t \in [0, T]$. On the other hand, by (3.30), we get

$$\begin{aligned} \dot{\tilde{\mu}}_t(x) &- (\bar{\Lambda}^*_{\tilde{\mu}_t, \tilde{m}_t} \tilde{\mu}_t)(x) \\ &+ \tilde{\mu}_t(x) \sum_{\substack{x' \in \mathcal{X}: \\ (x,x') \in \mathcal{E}_{\mathcal{X}}}} (\exp\{\hat{\alpha}_t(\tilde{\mu}_t)(x') - \hat{\alpha}_t(\tilde{\mu}_t)(x)\} - 1) \bar{\lambda}_{x,x'}(\tilde{\mu}_t, \tilde{m}_t) \\ &- \sum_{\substack{x_0 \in \mathcal{X}: \\ (x_0,x) \in \mathcal{E}_{\mathcal{X}}}} \tilde{\mu}_t(x_0)(\exp\{\hat{\alpha}_t(\tilde{\mu}_t)(x) - \hat{\alpha}_t(\tilde{\mu}_t)(x_0)\} - 1) \bar{\lambda}_{x_0,x}(\tilde{\mu}_t, \tilde{m}_t) = 0, \ \forall x \in \mathcal{X}, \end{aligned} \tag{3.34}$$

for almost all $t \in [0, T]$. Note that, by the optimality condition (3.28) and by (3.32), the mapping

$$\begin{aligned} u \mapsto \Bigg(&(\bar{\Lambda}^*_{u, \tilde{m}_t} u)(x) + u(x) \sum_{\substack{x' \in \mathcal{X}: \\ (x,x') \in \mathcal{E}_{\mathcal{X}}}} (\exp\{\hat{\alpha}_t(u)(x') - \hat{\alpha}_t(u)(x)\} - 1) \bar{\lambda}_{x,x'}(u, \tilde{m}_t) \\ &- \sum_{\substack{x_0 \in \mathcal{X}: \\ (x_0,x) \in \mathcal{E}_{\mathcal{X}}}} u(x_0)(\exp\{\hat{\alpha}_t(u)(x) - \hat{\alpha}_t(u)(x_0)\} - 1) \bar{\lambda}_{x_0,x}(u, \tilde{m}_t), \ x \in \mathcal{X} \Bigg) \in \mathbb{R}^{\mathcal{X}} \end{aligned}$$

on $\{u \in \mathcal{M}_1(\mathcal{X}) : u(x) \geq \delta/2 \, \forall x \in \mathcal{X}\}$ is identically equal to $\dot{\hat{\mu}}_t$ for almost all $t \in [0, T]$. Hence, by (3.33) and (3.34), and noting that $\tilde{\mu}_0 = \hat{\mu}_0 = \nu$, Gronwall inequality implies that $\tilde{\mu}_t = \hat{\mu}_t$ for all $t \in [0, T]$.

We have thus shown that $(\tilde{\mu}, \tilde{\theta}) = (\hat{\mu}, \hat{\theta})$, and the second equality in (3.30) implies that $\tilde{I}(\hat{\mu}, \hat{\theta}) = I^*(\hat{\mu}, \hat{\theta})$. This completes the proof of the theorem. □

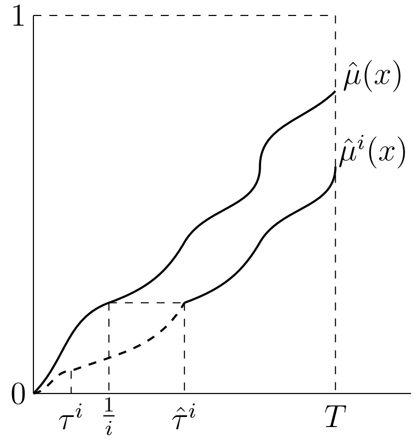

Figure 3.1: Figure depicting the idea of construction of $\hat{\mu}^i$ in the proof of Lemma 3.3.

3.6 Approximating the subsequential rate function

Let $\tilde{I} : D([0,T], \mathcal{M}_1(\mathcal{X})) \times D_\uparrow([0,T], \mathcal{M}(\mathcal{Y})) \to [0,+\infty]$ be a subsequential rate function for the family $\{(\mu^N, \theta^N), N \geq 1\}$, and suppose that, for some $\nu \in \mathcal{M}_1(\mathcal{X})$, $\tilde{I}(\mu,\theta) = +\infty$ unless $\mu_0 = \nu$. In this section, we show that $\tilde{I}(\mu,\theta) = I^*(\mu,\theta)$ for all $(\mu,\theta) \in D([0,T], \mathcal{M}_1(\mathcal{X})) \times D_\uparrow([0,T], \mathcal{M}(\mathcal{Y}))$. We shall proceed through a sequence of lemmas. In each lemma, we shall extend the conclusion $\tilde{I}(\mu,\theta) = I^*(\mu,\theta)$ to a larger class of elements (μ,θ) by producing a sequence (μ^i,θ^i) such that $\tilde{I}(\mu^i,\theta^i) = I^*(\mu^i,\theta^i)$ for all $i \geq 1$, $(\mu^i,\theta^i) \to (\mu,\theta)$ in $D([0,T], \mathcal{M}_1(\mathcal{X})) \times D_\uparrow([0,T], \mathcal{M}(\mathcal{Y}))$ as $i \to \infty$, and $I^*(\mu^i,\theta^i) \to I^*(\mu,\theta)$ as $i \to \infty$. Using these approximations, we finally show that $\tilde{I}(\mu,\theta) = I^*(\mu,\theta)$ for all $(\mu,\theta) \in D([0,T], \mathcal{M}_1(\mathcal{X})) \times D_\uparrow([0,T], \mathcal{M}(\mathcal{Y}))$ (see Theorem 3.9).

We start with an extension of the conclusion of Theorem 3.8 to all initial conditions ν.

Lemma 3.3. *Let $\nu \in \mathcal{M}_1(\mathcal{X})$ and let $\tilde{I} : D([0,T], \mathcal{M}_1(\mathcal{X})) \times D_\uparrow([0,T], \mathcal{M}(\mathcal{Y})) \to [0,+\infty]$ be a subsequential rate function such that $\tilde{I}(\mu,\theta) = +\infty$ unless $\mu_0 = \nu$. Suppose that $(\hat{\mu},\hat{\theta}) \in D([0,T], \mathcal{M}_1(\mathcal{X})) \times D_\uparrow([0,T], \mathcal{M}(\mathcal{Y}))$ is such that*

- $I^*(\hat{\mu},\hat{\theta}) < +\infty$,
- $\inf_{t\in[\delta,T]} \min_{x\in\mathcal{X}} \hat{\mu}_t(x) > 0$ *for all* $\delta > 0$,
- *the mapping* $[0,T] \ni t \mapsto \hat{\mu}_t \in \mathcal{M}_1(\mathcal{X})$ *is Lipschitz continuous,*
- $\hat{\theta}$, *when viewed as a measure on* $[0,T] \times \mathcal{Y}$, *admits the representation* $\hat{\theta}(dydt) = \hat{m}_t(dy)dt$ *for some* $\hat{m}_t \in \mathcal{M}_1(\mathcal{Y})$ *for almost all* $t \in [0,T]$, *and* $\inf_{t\in[0,T]} \min_{y\in\mathcal{Y}} \hat{m}_t(y) > 0$.

Then $\tilde{I}(\hat{\mu},\hat{\theta}) = I^*(\hat{\mu},\hat{\theta})$.

Proof. We begin with some notations. Let $\mathcal{X}_0 = \{x \in \mathcal{X} : \hat{\mu}_0(x) = 0\}$. For each $x \in \mathcal{X}_0$, let $\{x^x_k, 1 \le k \le l(x)\}$ be such that $\hat{\mu}_0(x^x_1) \ge 1/|\mathcal{X}_0|$ (in particular, $x^x_1 \notin \mathcal{X}_0$), $(x^x_k, x^x_{k+1}) \in \mathcal{E}_{\mathcal{X}}$ for all $1 \le k \le l(x) - 1$, and $(x^x_{l(x)}, x) \in \mathcal{E}_{\mathcal{X}}$, i.e., the collection of edges $\{(x^x_k, x^x_{k+1}), 1 \le k \le l(x) - 1\} \cup (x^x_{l(x)}, x)$ form a directed path of length $l(x)$ from x^x_1 to x. Also, for the given $\nu \in \mathcal{M}_1(\mathcal{X})$, let $\mu(\nu, \hat{\theta}) \in D([0, \infty), \mathcal{M}_1(\mathcal{X}))$ denote the unique solution to the ODE $\dot{\mu}_t = \bar{\Lambda}^*_{\mu_t, \hat{m}_t} \mu_t$ with initial condition $\mu_0 = \nu$.

For each $i \ge 1$, we define a path $\hat{\mu}^i \in D([0, T], \mathcal{M}_1(\mathcal{X}))$ as follows. Define $\hat{\mu}^i_t = \mu_t(\hat{\mu}_0, \hat{\theta})$ for $t \in [0, \tau^i]$ where $\tau^i = \inf\{t > 0 : \mu_t(\hat{\mu}_0, \hat{\theta})(x) = \hat{\mu}_{1/i}(x)/2$ for some $x \in \mathcal{X}_0\}$. Note that $\tau^i < +\infty$ for i sufficiently large. Also note that $\hat{\mu}^i_{\tau^i}(x) > 0$ for all $x \in \mathcal{X}$, and that the supremum over α_t in the definition of $I^*(\hat{\mu}^i, \hat{\theta})$ (see (3.8)) is attained at $\alpha_t = 0$ for all $t \in [0, \tau^i]$. Let $\varepsilon_i(x) = \hat{\mu}_{1/i}(x) - \hat{\mu}^i_{\tau^i}(x)$ for $x \in \mathcal{X}$ and $i \ge 1$. Since the mapping $t \mapsto \hat{\mu}_t$ is Lipschitz continuous, we see that $\tau^i \to 0$ as $i \to \infty$, and $\varepsilon_i(x) \to 0$ as $i \to \infty$ for all $x \in \mathcal{X}$. For each $x \in \tilde{\mathcal{X}}_0 \coloneqq \mathcal{X}_0 \cap \{x \in \mathcal{X}_0 : \varepsilon_i(x) > 0\}$, we shall now move the mass $\varepsilon_i(x)$ from the vertex x^x_1 to x via the edges defined in the previous paragraph using a piecewise constant velocity path. Denote the elements of $\tilde{\mathcal{X}}_0$ by $x_1, x_2 \ldots, x_{|\tilde{\mathcal{X}}_0|}$, let $l(x_0) = 0$ and $\varepsilon_i(x_0) = 0$. Given $r \in \{0, 1, \ldots, |\tilde{\mathcal{X}}_0| - 1\}$, $s \in \{0, 1, \ldots, l(x_{r+1}) - 1\}$, and $t \in [\tau^i + \sum_{m=0}^r l(x_m)\varepsilon_i(x_m) + s\varepsilon_i(x_{r+1}), \tau^i + \sum_{m=0}^r l(x_m)\varepsilon_i(x_m) + (s+1)\varepsilon_i(x_{r+1}))$, define

$$\dot{\hat{\mu}}^i_t(x) \coloneqq \begin{cases} -1 & \text{if } x = x^{x_{r+1}}_{s+1} \\ 1 & \text{if } x = x^{x_{r+1}}_{s+2} \\ 0 & \text{otherwise,} \end{cases}$$

i.e., we transport a mass of $\varepsilon_i(x_{r+1})$ at unit rate from the node $x^{x_{r+1}}_{s+1}$ to $x^{x_{r+1}}_{s+2}$ during the above time interval. Note that we have $\hat{\mu}^i_t(x) = \hat{\mu}_t(x)$ for all $x \in \tilde{\mathcal{X}}_0$ at time $t = \tau^i + \sum_{m=1}^{|\tilde{\mathcal{X}}_0|} l(x_m)\varepsilon_i(x_m)$. Similarly, for $x \in \mathcal{X} \setminus \tilde{\mathcal{X}}_0$ with $\varepsilon_i(x) > 0$, one defines a sequence of edges from a suitable $x' \in \mathcal{X} \setminus \tilde{\mathcal{X}}_0$ (possibly from multiple $x' \in \mathcal{X} \setminus \tilde{\mathcal{X}}_0$) with $\varepsilon_i(x') < 0$ and moves the mass $\varepsilon_i(x)$ to x through similar piecewise constant velocity trajectories defined above. For each $x \in \mathcal{X} \setminus \tilde{\mathcal{X}}_0$ with $\varepsilon_i(x) < 0$, we similarly move the mass $\varepsilon_i(x)$ from x to suitable vertices in $\mathcal{X} \setminus \tilde{\mathcal{X}}_0$ via piecewise constant velocity trajectories. At the end of this procedure, we have $\hat{\mu}^i_{\hat{\tau}^i} = \hat{\mu}_{1/i}$ for some $\hat{\tau}^i \ge \tau^i$. We now define $\hat{\mu}^i_t = \hat{\mu}_{t+1/i-\hat{\tau}^i}$ for all $t \in [\hat{\tau}^i, T]$ (see Figure 3.1 for a pictorial representation of $\hat{\mu}^i$). Since $\varepsilon_i(x) \to 0$ as $i \to \infty$ for all $x \in \mathcal{X}$, we have that $\hat{\tau}^i \to 0$ as $i \to \infty$.

Also, for each $i \ge 1$ and $t \in [0, T]$, define the probability measure $\hat{m}^i_t$ on $\mathcal{Y}$ by

$$\hat{m}^i_t(y) \coloneqq \begin{cases} \hat{m}_t(y) & \text{if } t \in [0, \tau^i], \\ \hat{m}_{\tau^i}(y) & \text{if } t \in [\tau^i, \hat{\tau}^i], \\ \hat{m}_{t+1/i-\hat{\tau}^i}(y) & \text{if } t \in (\hat{\tau}^i, T], \end{cases}$$

for all $y \in \mathcal{Y}$, and define the measure $\hat{\theta}^i$ on $[0,T] \times \mathcal{Y}$ by $\hat{\theta}^i(dydt) = \hat{m}^i_t(dy)dt$. Clearly, $\hat{\theta}^i \in D_\uparrow([0,T], \mathcal{M}(\mathcal{Y}))$.

Thanks to the fact that $\hat{\mu}^i_{\tau^i}(x) > 0$ for all $x \in \mathcal{X}$ and the fact that $\alpha_t = 0$ attains the supremum in the definition of $I^*(\hat{\mu}^i, \hat{\theta}^i)$ for all $t \in [0, \tau^i]$, using arguments similar to those used in the proof of Theorem 3.8, one can now construct a bounded measurable function $\hat{\alpha}^i : [0,T] \times \mathcal{M}_1(\mathcal{X}) \to \mathbb{R}^{\mathcal{X}}$ such that $\hat{\alpha}^i_t(\hat{\mu}^i_t)$ attains the supremum over α_t in the definition of $I^*(\hat{\mu}^i, \hat{\theta}^i)$ (in (3.8)) and $\hat{\alpha}^i_t(\cdot)$ is continuous on $\mathcal{M}_1(\mathcal{X})$ for all $t \in [0,T]$. Similarly, since $\hat{\theta}^i$ satisfies the conditions of Theorem 3.8, one can construct a bounded measurable function $\hat{g}^i : [0,T] \times \mathcal{M}_1(\mathcal{X}) \times \mathcal{Y} \to \mathbb{R}$ such that $\hat{g}^i_t(\hat{\mu}^i_t, \cdot)$ attains the supremum over g_t in the definition of $I^*(\hat{\mu}^i, \hat{\theta}^i)$ and $\hat{g}^i_t(\cdot)$ is continuous on $\mathcal{M}_1(\mathcal{X})$ for each $t \in [0,T]$. Hence, using arguments similar to those used in the proof of Theorem 3.8, one concludes that $\tilde{I}(\hat{\mu}^i, \hat{\theta}^i) = I^*(\hat{\mu}^i, \hat{\theta}^i)$ for all $i \geq 1$.

Let us now show that $I^*(\hat{\mu}^i, \hat{\theta}^i) \to I^*(\hat{\mu}, \hat{\theta})$ as $i \to \infty$. For the fast component, since $\hat{\tau}^i \to 0$, we see that $\hat{\theta}^i \to \hat{\theta}$ in $D_\uparrow([0,T], \mathcal{M}(\mathcal{Y}))$ as $i \to \infty$. By assumption (D2), we see that

$$0 \leq \sup_{i \geq 1, t \in [0,T]} \left\{ \sup_{g_t \in \mathbb{R}^{\mathcal{Y}}} - \int_{\mathcal{Y}} \left(L_{\hat{\mu}^i_t} g_t(\cdot)(y) \right. \right.$$
$$\left. \left. + \int_{\mathcal{E}_{\mathcal{Y}}} \tau(Dg_t(y,\Delta)) \gamma_{y,y+d\Delta}(\hat{\mu}^i_t) \right) \hat{m}^i_t(dy) \right\} < +\infty,$$

and hence the bounded convergence theorem immediately yields

$$\int_{[0,\hat{\tau}^i]} \sup_{g_t \in B(\mathcal{Y})} \left\{ - \int_{\mathcal{Y}} \left(L_{\hat{\mu}^i_t} g_t(\cdot)(y) + \int_{\mathcal{E}_{\mathcal{Y}}} \tau(Dg_t(y,\Delta)) \gamma_{y,y+d\Delta}(\hat{\mu}^i_t) \right) \hat{m}^i_t(dy) \right\} dt \to 0$$

and

$$\int_{[T+1/i-\hat{\tau}^i, T]} \sup_{g_t \in B(\mathcal{Y})} \left\{ - \int_{\mathcal{Y}} \left(L_{\hat{\mu}_t} g_t(\cdot)(y) + \int_{\mathcal{E}_{\mathcal{Y}}} \tau(Dg_t(y,\Delta)) \gamma_{y,y+d\Delta}(\hat{\mu}_t) \right) \hat{m}_t(dy) \right\} dt \to 0$$

as $i \to \infty$. Noting that $\hat{m}^i_t = \hat{m}_{t+1/i-\hat{\tau}^i}$ and $\hat{\mu}^i_t = \hat{\mu}_{t+1/i-\hat{\tau}^i}$ for all $t \in [\hat{\tau}^i, T]$, the above convergences imply that

$$\int_{[0,T]} \sup_{g_t \in B(\mathcal{Y})} \left\{ - \int_{\mathcal{Y}} \left(L_{\hat{\mu}^i_t} g_t(\cdot)(y) + \int_{\mathcal{E}_{\mathcal{Y}}} \tau(Dg_t(y,\Delta)) \gamma_{y,y+d\Delta}(\hat{\mu}^i_t) \right) \hat{m}^i_t(dy) \right\} dt$$
$$\to \int_{[0,T]} \sup_{g_t \in B(\mathcal{Y})} \left\{ - \int_{\mathcal{Y}} \left(L_{\hat{\mu}_t} g_t(\cdot)(y) + \int_{\mathcal{E}_{\mathcal{Y}}} \tau(Dg_t(y,\Delta)) \gamma_{y,y+d\Delta}(\hat{\mu}_t) \right) \hat{m}_t(dy) \right\} dt$$

as $i \to \infty$.

For the slow component, since $\hat{\tau}^i \to 0$ as $i \to \infty$, using the absolute continuity of the mapping $t \mapsto \hat{\mu}_t$ and the definition of the paths $\hat{\mu}^i$, it follows from the dominated convergence theorem that $\hat{\mu}^i_t \to \hat{\mu}_t$ as $i \to \infty$ uniformly in $t \in [0,T]$ and hence we have that $\hat{\mu}^i \to \hat{\mu}$ in $D([0,T], \mathcal{M}_1(\mathcal{X}))$ as $i \to \infty$. Let us first show that

$$\int_{[0,\hat{\tau}^i]} \sup_{\alpha_t \in \mathbb{R}^{\mathcal{X}}} \left\{ \langle \alpha_t, \dot{\hat{\mu}}^i_t - \bar{\Lambda}^*_{\hat{\mu}^i_t, \hat{m}^i_t} \hat{\mu}^i_t \rangle - \int_{\mathcal{X}\times\mathcal{E}_{\mathcal{X}}} \tau(D\alpha_t(x,\Delta)) \bar{\lambda}_{x,x+d\Delta}(\hat{\mu}^i_t, \hat{m}^i_t) \hat{\mu}^i_t(dx) \right\} dt$$

converges to 0 as $i \to \infty$. Towards this, let $t \in [\tau^i + \sum_{m=0}^{r} l(x_m)\varepsilon_i(x_m) + s\varepsilon_i(x_{r+1}), \tau^i + \sum_{m=0}^{r} l(x_m)\varepsilon_i(x_m) + (s+1)\varepsilon_i(x_{r+1}))$ where $r \in \{0,1,\ldots,|\tilde{\mathcal{X}}_0|-1\}$, and $s \in \{0,1,\ldots,l(x_{r+1})-1\}$. Note that, we have

$$\begin{aligned}
&\sup_{\alpha_t \in \mathbb{R}^{\mathcal{X}}} \left\{ \langle \alpha_t, \dot{\hat{\mu}}^i_t - \bar{\Lambda}^*_{\hat{\mu}^i_t, \hat{m}^i_t} \hat{\mu}^i_t \rangle - \int_{\mathcal{X}\times\mathcal{E}_{\mathcal{X}}} \tau(D\alpha_t(x,\Delta)) \bar{\lambda}_{x,x+d\Delta}(\hat{\mu}^i_t, \hat{m}^i_t) \hat{\mu}^i_t(dx) \right\} \\
&\quad \le \sup_{\alpha_t \in \mathbb{R}^{\mathcal{X}}} \Big((\alpha_t(x^{x_{r+1}}_{s+2}) - \alpha_t(x^{x_{r+1}}_{s+1})) - (\exp\{\alpha_t(x^{x_{r+1}}_{s+2}) - \alpha_t(x^{x_{r+1}}_{s+1})\} - 1) \\
&\qquad\qquad \times \bar{\lambda}_{x^{x_{r+1}}_{s+1}, x^{x_{r+1}}_{s+2}}(\hat{\mu}^i_t, \hat{m}^i_t) \hat{\mu}^i_t(x^{x_{r+1}}_{s+1}) \Big) \\
&\qquad - \inf_{\alpha_t \in \mathbb{R}^{\mathcal{X}}} \sum_{\substack{(x,x')\in\mathcal{E}_{\mathcal{X}}: \\ (x,x') \neq (x^{x_{r+1}}_{s+1}, x^{x_{r+1}}_{s+2})}} (\exp\{\alpha_t(x') - \alpha_t(x)\} - 1) \bar{\lambda}_{x,x'}(\hat{\mu}^i_t, \hat{m}^i_t) \hat{\mu}^i_t(x) \\
&\quad \le \log \frac{1}{c\hat{\mu}^i_t(x^{x_{r+1}}_{s+1})} + c_1
\end{aligned}$$

where $c = \min_{(x,x')\in\mathcal{E}_{\mathcal{X}}} \min_{y\in\mathcal{Y}} \min_{\xi\in\mathcal{M}_1(\mathcal{X})} \lambda_{x,x'}(\xi, y)$ and $c_1 > 0$ is a suitable constant to bound the extra additive terms. Hence, using a variable change $u = c\hat{\mu}^i_t(x^{x_{r+1}}_{s+1})$, we see that

$$\begin{aligned}
&\int \sup_{\alpha_t \in \mathbb{R}^{\mathcal{X}}} \left\{ \langle \alpha_t, \dot{\hat{\mu}}^i_t - \bar{\Lambda}^*_{\hat{\mu}^i_t, \hat{m}^i_t} \hat{\mu}^i_t \rangle - \int_{\mathcal{X}\times\mathcal{E}_{\mathcal{X}}} \tau(D\alpha_t(x,\Delta)) \bar{\lambda}_{x,x+d\Delta}(\hat{\mu}^i_t, \hat{m}^i_t) \hat{\mu}^i_t(dx) \right\} dt \\
&\quad \le -\frac{1}{c}(u\log u - u)\Big|_{c\hat{\mu}^i_{t_1}(x^{x_{r+1}}_{s+1})}^{c\hat{\mu}^i_{t_2}(x^{x_{r+1}}_{s+1})} + c_1\varepsilon_i(x_{r+1}) \\
&\quad = o(1)
\end{aligned}$$

as $i \to \infty$, where $t_1 = \tau^i + \sum_{m=0}^{r} l(x_m)\varepsilon_i(x_m) + s\varepsilon_i(x_{r+1})$, $t_2 = t_1 + \varepsilon_i(x_{r+1})$ and the above integral is evaluated over the time interval $[\tau^i + \sum_{m=0}^{r} l(x_m)\varepsilon_i(x_m) + s\varepsilon_i(x_{r+1}), \tau^i + \sum_{m=0}^{r} l(x_m)\varepsilon_i(x_m) + (s+1)\varepsilon_i(x_{r+1}))$. Hence, repeating the above calculation for each constant velocity section of the path $\hat{\mu}^i$ during the time interval $[\tau^i, \hat{\tau}^i]$, we see that

$$\int_{[0,\hat{\tau}^i]} \sup_{\alpha_t \in \mathbb{R}^{\mathcal{X}}} \left\{ \langle \alpha_t, \dot{\hat{\mu}}^i_t - \bar{\Lambda}^*_{\hat{\mu}^i_t, \hat{m}^i_t} \hat{\mu}^i_t \rangle - \int_{\mathcal{X}\times\mathcal{E}_{\mathcal{X}}} \tau(D\alpha_t(x,\Delta)) \bar{\lambda}_{x,x+d\Delta}(\hat{\mu}^i_t, \hat{m}^i_t) \hat{\mu}^i_t(dx) \right\} dt$$

converges to 0 as $i \to \infty$. Therefore, noting that $\hat{\mu}^i_t = \hat{\mu}_{t+1/i-\hat{\tau}^i}$ and $\hat{m}^i_t = \hat{m}_{t+1/i-\hat{\tau}^i}$ for $t \in [\hat{\tau}^i, T]$, and $\hat{\mu}^i_t = \mu_t(\hat{\mu}_0, \hat{\theta})$ on $t \in [0, \tau^i]$, we have

$$\begin{aligned}
&\Bigg| \int_{[0,T]} \sup_{\alpha_t \in \mathbb{R}^{\mathcal{X}}} \left\{ \langle \alpha_t, \dot{\hat{\mu}}_t - \bar{\Lambda}^*_{\hat{\mu}_t, \hat{m}_t} \hat{\mu}_t \rangle - \int_{\mathcal{X} \times \mathcal{E}_{\mathcal{X}}} \tau(D\alpha_t(x, \Delta)) \bar{\lambda}_{x, x+d\Delta}(\hat{\mu}_t, \hat{m}_t) \hat{\mu}_t(dx) \right\} dt \\
&\qquad - \int_{[0,T]} \sup_{\alpha_t \in \mathbb{R}^{\mathcal{X}}} \left\{ \langle \alpha_t, \dot{\hat{\mu}}^i_t - \bar{\Lambda}^*_{\hat{\mu}^i_t, \hat{m}^i_t} \hat{\mu}^i_t \rangle - \int_{\mathcal{X} \times \mathcal{E}_{\mathcal{X}}} \tau(D\alpha_t(x, \Delta)) \bar{\lambda}_{x, x+d\Delta}(\hat{\mu}^i_t, \hat{m}^i_t) \hat{\mu}^i_t(dx) \right\} dt \Bigg| \\
&\leq \int_{[0,1/i]} \sup_{\alpha_t \in \mathbb{R}^{\mathcal{X}}} \left\{ \langle \alpha_t, \dot{\hat{\mu}}_t - \bar{\Lambda}^*_{\hat{\mu}_t, \hat{m}_t} \hat{\mu}_t \rangle - \int_{\mathcal{X} \times \mathcal{E}_{\mathcal{X}}} \tau(D\alpha_t(x, \Delta)) \bar{\lambda}_{x, x+d\Delta}(\hat{\mu}_t, \hat{m}_t) \hat{\mu}_t(dx) \right\} dt \\
&\qquad + \int_{[T+1/i-\hat{\tau}^i, T]} \sup_{\alpha_t \in \mathbb{R}^{\mathcal{X}}} \left\{ \langle \alpha_t, \dot{\hat{\mu}}_t - \bar{\Lambda}^*_{\hat{\mu}_t, \hat{m}_t} \hat{\mu}_t \rangle - \int_{\mathcal{X} \times \mathcal{E}_{\mathcal{X}}} \tau(D\alpha_t(x, \Delta)) \bar{\lambda}_{x, x+d\Delta}(\hat{\mu}_t, \hat{m}_t) \hat{\mu}_t(dx) \right\} dt \\
&\qquad + \int_{[0,\hat{\tau}^i]} \sup_{\alpha_t \in \mathbb{R}^{\mathcal{X}}} \left\{ \langle \alpha_t, \dot{\hat{\mu}}^i_t - \bar{\Lambda}^*_{\hat{\mu}^i_t, \hat{m}^i_t} \hat{\mu}^i_t \rangle - \int_{\mathcal{X} \times \mathcal{E}_{\mathcal{X}}} \tau(D\alpha_t(x, \Delta)) \bar{\lambda}_{x, x+d\Delta}(\hat{\mu}^i_t, \hat{m}^i_t) \hat{\mu}^i_t(dx) \right\} dt \\
&\to 0
\end{aligned}$$

as $i \to \infty$. We have thus shown that $I^*(\hat{\mu}^i, \hat{\theta}^i) \to I^*(\hat{\mu}, \hat{\theta})$ as $i \to \infty$.

Since $(\hat{\mu}^i, \hat{\theta}^i) \to (\hat{\mu}, \hat{\theta})$ in $D([0,T], \mathcal{M}_1(\mathcal{X})) \times D_\uparrow([0,T], \mathcal{M}(\mathcal{Y}))$ as $i \to \infty$, the lower semicontinuity of $\tilde{I}$ implies that $\liminf_{i\to\infty} \tilde{I}(\hat{\mu}^i, \hat{\theta}^i) \geq \tilde{I}(\hat{\mu}, \hat{\theta})$. Therefore, using the above convergence and the fact that $\tilde{I}(\hat{\mu}^i, \hat{\theta}^i) = I^*(\hat{\mu}^i, \hat{\theta}^i)$ for all $i \geq 1$, we see that $\tilde{I}(\hat{\mu}, \hat{\theta}) \leq I^*(\hat{\mu}, \hat{\theta})$. On the other hand, since $\tilde{I}(\hat{\mu}, \hat{\theta}) \geq I^*(\hat{\mu}, \hat{\theta})$, it follows that $\tilde{I}(\hat{\mu}, \hat{\theta}) = I^*(\hat{\mu}, \hat{\theta})$. This completes the proof of the lemma. □

Remark 3.2. We shall repeatedly use the immediately preceding argument; starting with an element $(\hat{\mu}, \hat{\theta}) \in D([0,T], \mathcal{M}_1(\mathcal{X})) \times D_\uparrow([0,T], \mathcal{M}(\mathcal{Y}))$, we shall produce a sequence $(\hat{\mu}^i, \hat{\theta}^i) \in D([0,T], \mathcal{M}_1(\mathcal{X})) \times D_\uparrow([0,T], \mathcal{M}(\mathcal{Y}))$, $i \geq 1$, such that $\tilde{I}(\hat{\mu}^i, \hat{\theta}^i) = I^*(\hat{\mu}^i, \hat{\theta}^i)$ for all $i \geq 1$, $(\hat{\mu}^i, \hat{\theta}^i) \to (\hat{\mu}, \hat{\theta})$ in $D([0,T], \mathcal{M}_1(\mathcal{X})) \times D_\uparrow([0,T], \mathcal{M}(\mathcal{Y}))$ as $i \to \infty$ and $I^*(\hat{\mu}^i, \hat{\theta}^i) \to I^*(\hat{\mu}, \hat{\theta})$ as $i \to \infty$, and use the above argument to conclude that $\tilde{I}(\hat{\mu}, \hat{\theta}) = I^*(\hat{\mu}, \hat{\theta})$.

We now extend the conclusion of the previous lemma to all elements $\hat{\theta} \in D_\uparrow([0,T], \mathcal{M}(\mathcal{Y}))$.

Lemma 3.4. *Let $\nu \in \mathcal{M}_1(\mathcal{X})$ and let $\tilde{I} : D([0,T], \mathcal{M}_1(\mathcal{X})) \times D_\uparrow([0,T], \mathcal{M}(\mathcal{Y})) \to [0, +\infty]$ be a subsequential rate function such that $\tilde{I}(\mu, \theta) = +\infty$ unless $\mu_0 = \nu$. Suppose that $(\hat{\mu}, \hat{\theta}) \in D([0,T], \mathcal{M}_1(\mathcal{X})) \times D_\uparrow([0,T], \mathcal{M}(\mathcal{Y}))$ is such that*

- $I^*(\hat{\mu}, \hat{\theta}) < +\infty$,
- $\inf_{t\in[\delta,T]} \min_{x\in\mathcal{X}} \hat{\mu}_t(x) > 0$ *for all* $\delta > 0$,
- *the mapping* $[0,T] \ni t \mapsto \hat{\mu}_t \in \mathcal{M}_1(\mathcal{X})$ *is Lipschitz continuous.*

Then $\tilde{I}(\hat{\mu},\hat{\theta}) = I^*(\hat{\mu},\hat{\theta})$.

Proof. Let $\hat{\theta}$, when viewed as a measure on $[0,T]\times\mathcal{Y}$, admit the representation $\hat{\theta}(dydt) = \hat{m}_t(dy)dt$, where $\hat{m}_t \in \mathcal{M}_1(\mathcal{Y})$ for almost all $t\in[0,T]$. For each $i\geq 1$ and for each $t\in[0,T]$, define the probability measure $\hat{m}_t^i$ on $\mathcal{Y}$ by

$$\hat{m}_t^i(y) = \frac{\hat{m}_t(y)+1/i}{1+|\mathcal{Y}|/i}, y\in\mathcal{Y}, \tag{3.35}$$

and, for each $i\geq 1$, define the measure $\hat{\theta}^i(dydt)$ on $[0,T]\times\mathcal{M}(\mathcal{Y})$ by $\hat{\theta}^i(dydt) \coloneqq \hat{m}_t^i(dy)dt$. Clearly, $\hat{\theta}^i \in D_\uparrow([0,T],\mathcal{M}(\mathcal{Y}))$ for all $i\geq 1$, and $\hat{\theta}^i\to\hat{\theta}$ in $D_\uparrow([0,T],\mathcal{M}(\mathcal{Y}))$ as $i\to\infty$. Since $(\hat{\mu},\hat{\theta}^i)$ satisfies the assumptions of Lemma 3.3, we have $\tilde{I}(\hat{\mu},\hat{\theta}^i) = I^*(\hat{\mu},\hat{\theta}^i)$.

Since, for each $t\in[0,T]$, the mapping

$$(g_t,m_t)\mapsto \max\Big\{-\int_{\mathcal{Y}}\Big(L_{\hat{\mu}_t}g_t(\cdot)(y)+\int_{\mathcal{E}_\mathcal{Y}}\tau(Dg_t(y,\Delta))\gamma_{y,y+d\Delta}(\hat{\mu}_t)\Big)m_t(dy),0\Big\}$$

on $(\mathbb{R}\cup\{+\infty,-\infty\})^{\mathcal{Y}}\times\mathcal{M}_1(\mathcal{Y})$ is bounded and continuous (thanks to assumption (D2)), by an application of the Berge's maximum theorem, it follows that the mapping

$$m_t\mapsto \sup_{g_t\in\mathbb{R}^{\mathcal{Y}}} -\int_{\mathcal{Y}}\Big(L_{\hat{\mu}_t}g_t(\cdot)(y)+\int_{\mathcal{E}_\mathcal{Y}}\tau(Dg_t(y,\Delta))\gamma_{y,y+d\Delta}(\hat{\mu}_t)\Big)m_t(dy) \tag{3.36}$$

is continuous on $\mathcal{M}_1(\mathcal{Y})$. Similarly, for each $t\geq 0$, by assumption (C2), it follows that the mapping

$$(\alpha_t,m_t)\mapsto\langle\alpha_t,\dot{\hat{\mu}}_t-\bar{\Lambda}^*_{\hat{\mu}_t,m_t}\rangle-\int_{\mathcal{X}\times\mathcal{E}_\mathcal{X}}\tau(D\alpha_t(x,\Delta))\bar{\lambda}_{x.x+d\Delta}(\hat{\mu}_t,m_t)\hat{\mu}_t(dx)$$

is bounded and continuous on $\mathbb{R}^{\mathcal{X}}\times\mathcal{M}_1(\mathcal{Y})$. Again, by the Berge's maximum theorem,

$$m_t\mapsto\sup_{\alpha_t\in\mathbb{R}^{\mathcal{X}}}\Big\{\langle\alpha_t,\dot{\hat{\mu}}_t-\bar{\Lambda}^*_{\hat{\mu}_t,m_t}\rangle-\int_{\mathcal{X}\times\mathcal{E}_\mathcal{X}}\tau(D\alpha_t(x,\Delta))\bar{\lambda}_{x,x+d\Delta}(\hat{\mu}_t,m_t)\hat{\mu}_t(dx)\Big\}$$

is continuous on $\mathcal{M}_1(\mathcal{Y})$. Therefore, for each $t\in[0,T]$, we see that

$$\begin{aligned}\sup_{\alpha_t\in\mathbb{R}^{\mathcal{X}}}&\Big\{\langle\alpha_t,\dot{\hat{\mu}}_t-\bar{\Lambda}^*_{\hat{\mu}_t,\hat{m}_t^i}\rangle-\int_{\mathcal{X}\times\mathcal{E}_\mathcal{X}}\tau(D\alpha_t(x,\Delta))\bar{\lambda}_{x,x+d\Delta}(\hat{\mu}_t,\hat{m}_t^i)\hat{\mu}_t(dx)\Big\}\\ &\to\sup_{\alpha_t\in\mathbb{R}^{\mathcal{X}}}\Big\{\langle\alpha_t,\dot{\hat{\mu}}_t-\bar{\Lambda}^*_{\hat{\mu}_t,\hat{m}_t}\rangle-\int_{\mathcal{X}\times\mathcal{E}_\mathcal{X}}\tau(D\alpha_t(x,\Delta))\bar{\lambda}_{x,x+d\Delta}(\hat{\mu}_t,\hat{m}_t)\hat{\mu}_t(dx)\Big\},\end{aligned}$$

and

$$\sup_{g_t\in B(\mathcal{Y})} -\int_{\mathcal{Y}}\Big(L_{\hat\mu_t}g_t(\cdot)(y)+\int_{\mathcal{E}_{\mathcal{Y}}}\tau(Dg_t(y,\Delta))\gamma_{y,y+d\Delta}(\hat\mu_t)\Big)\hat m_t^i(dy)$$
$$\to \sup_{g_t\in B(\mathcal{Y})} -\int_{\mathcal{Y}}\Big(L_{\hat\mu_t}g_t(\cdot)(y)+\int_{\mathcal{E}_{\mathcal{Y}}}\tau(Dg_t(y,\Delta))\gamma_{y,y+d\Delta}(\hat\mu_t)\Big)\hat m_t(dy)$$

as $i\to\infty$. Noting that

$$0\le \sup_{i\ge 1,t\in[0,T]}\sup_{\alpha_t\in\mathbb{R}^{\mathcal{X}}}\Big\{\langle\alpha_t,\dot{\hat\mu}_t-\bar\Lambda^*_{\hat\mu_t,\hat m_t^i}\rangle - \int_{\mathcal{X}\times\mathcal{E}_{\mathcal{X}}}\tau(D\alpha_t(x,\Delta))\bar\lambda_{x,x+d\Delta}(\hat\mu_t,\hat m_t^i)\hat\mu_t(dx)\Big\}<+\infty$$

and

$$0\le \sup_{i\ge 1,t\in[0,T]}\sup_{g_t\in\mathbb{R}^{\mathcal{Y}}}\Big\{-\int_{\mathcal{Y}}\Big(L_{\hat\mu_t}g_t(\cdot)(y) + \int_{\mathcal{E}_{\mathcal{Y}}}\tau(Dg_t(y,\Delta))\gamma_{y,y+d\Delta}(\hat\mu_t)\Big)\hat m_t^i(dy)\Big\}<+\infty,$$

using the bounded convergence theorem, we obtain that $I^*(\hat\mu,\hat\theta^i)\to I^*(\hat\mu,\hat\theta)$ as $i\to\infty$. Thanks to Remark 3.2, this completes the proof of the lemma. □

We now extend the conclusion of the previous lemma to the case when the mapping $[0,T]\ni t\mapsto\mu_t\in\mathcal{M}_1(\mathcal{X})$ is not necessarily Lipschitz continuous.

Lemma 3.5. *Let $\nu\in\mathcal{M}_1(\mathcal{X})$ and let $\tilde I : D([0,T],\mathcal{M}_1(\mathcal{X}))\times D_\uparrow([0,T],\mathcal{M}(\mathcal{Y}))\to[0,+\infty]$ be a subsequential rate function such that $\tilde I(\mu,\theta)=+\infty$ unless $\mu_0=\nu$. Suppose that $(\hat\mu,\hat\theta)\in D([0,T],\mathcal{M}_1(\mathcal{X}))\times D_\uparrow([0,T],\mathcal{M}(\mathcal{Y}))$ is such that $I^*(\hat\mu,\hat\theta)<+\infty$, and $\inf_{t\in[\delta,T]}\min_{x\in\mathcal{X}}\hat\mu_t(x)>0$ for all $\delta>0$. Then $\tilde I(\hat\mu,\hat\theta)=I^*(\hat\mu,\hat\theta)$.*

Proof. Let us first suppose that the mapping $t\mapsto\hat\mu_t$ is locally Lipschitz continuous at $t=0$ so that $\sup_{t\in[0,\eta]}\|\dot{\hat\mu}_t\|<+\infty$ for some $\eta>0$. Define a sequence of paths $\hat\mu^i$, $i\ge1$, by $\hat\mu_0^i=\hat\mu_0$, and

$$\dot{\hat\mu}_t^i=\dot{\hat\mu}_t\mathbf{1}_{\{\|\dot{\hat\mu}_t\|\le i\}}+\bar\Lambda^*_{\hat\mu_t^i,\hat m_t}\hat\mu_t^i\mathbf{1}_{\{\|\dot{\hat\mu}_t\|>i\}},\ t\in[0,T].$$

Since $I^*(\hat\mu,\hat\theta)<+\infty$, by Lemma 3.1, it follows that the mapping $t\mapsto\hat\mu_t$ is absolutely continuous and by the dominated convergence theorem one easily concludes that $\hat\mu_t^i\to\hat\mu_t$ as $i\to\infty$ uniformly in $t\in[0,T]$. Thus, by the assumption $\inf_{t\in[\delta,T]}\min_{x\in\mathcal{X}}\hat\mu_t(x)>0$ for all $\delta>0$, it follows that $\hat{\mu}^i \in D([0,T], \mathcal{M}_1(\mathcal{X}))$ for all i sufficiently large. Note that $(\hat{\mu}^i, \hat{\theta})$ satisfies the conditions of Lemma 3.4 and hence $\tilde{I}(\hat{\mu}^i, \hat{\theta}) = I^*(\hat{\mu}^i, \hat{\theta})$ for all $i \geq 1$, that $\hat{\mu}^i \to \hat{\mu}$ in $D([0,T], \mathcal{M}_1(\mathcal{X}))$ as $i \to \infty$, and that $\hat{\mu}^i_t = \hat{\mu}_t$ for all $t \in [0, \eta]$ for all sufficiently large i.

Let us now show that $I^*(\hat{\mu}^i, \hat{\theta}) \to I^*(\hat{\mu}^i, \hat{\theta})$ as $i \to \infty$. By the arguments similar to those used in the proof of Lemma 3.4, using Berge's maximum theorem, for each $t \in [0,T]$, the mapping

$$u \mapsto \sup_{g_t \in B(\mathcal{Y})} - \int_{\mathcal{Y}} \left(L_u g_t(\cdot)(y) + \int_{\mathcal{E}_{\mathcal{Y}}} \tau(Dg_t(y, \Delta)) \gamma_{y,y+d\Delta}(u) \right) \hat{m}_t(dy)$$

is continuous on $\mathcal{M}_1(\mathcal{X})$, and hence

$$\sup_{g_t \in B(\mathcal{Y})} - \int_{\mathcal{Y}} \left(L_{\hat{\mu}^i_t} g_t(\cdot)(y) + \int_{\mathcal{E}_{\mathcal{Y}}} \tau(Dg_t(y, \Delta)) \gamma_{y,y+d\Delta}(\hat{\mu}^i_t) \right) \hat{m}_t(dy)$$
$$\to \sup_{g_t \in B(\mathcal{Y})} - \int_{\mathcal{Y}} \left(L_{\hat{\mu}_t} g_t(\cdot)(y) + \int_{\mathcal{E}_{\mathcal{Y}}} \tau(Dg_t(y, \Delta)) \gamma_{y,y+d\Delta}(\hat{\mu}_t) \right) \hat{m}_t(dy)$$

as $i \to \infty$. Therefore, by the bounded convergence theorem, we have

$$\int_{[0,T]} \sup_{g_t \in B(\mathcal{Y})} \left\{ - \int_{\mathcal{Y}} \left(L_{\hat{\mu}^i_t} g_t(\cdot)(y) + \int_{\mathcal{E}_{\mathcal{Y}}} \tau(Dg_t(y, \Delta)) \gamma_{y,y+d\Delta}(\hat{\mu}^i_t) \right) \hat{m}_t(dy) \right\} dt$$
$$\to \int_{[0,T]} \sup_{g_t \in B(\mathcal{Y})} \left\{ - \int_{\mathcal{Y}} \left(L_{\hat{\mu}_t} g_t(\cdot)(y) + \int_{\mathcal{E}_{\mathcal{Y}}} \tau(Dg_t(y, \Delta)) \gamma_{y,y+d\Delta}(\hat{\mu}_t) \right) \hat{m}_t(dy) \right\} dt$$

as $i \to \infty$.

For the slow component, define

$$Z^i_t := \sup_{\alpha_t \in \mathbb{R}^{\mathcal{X}}} \left\{ \langle \alpha_t, \dot{\hat{\mu}}^i_t - \bar{\Lambda}^*_{\hat{\mu}^i_t, \hat{m}_t} \hat{\mu}^i_t \rangle - \int_{\mathcal{X} \times \mathcal{E}_{\mathcal{X}}} \tau(D\alpha_t(x, \Delta)) \bar{\lambda}_{x,x+d\Delta}(\hat{\mu}^i_t, \hat{m}_t) \hat{\mu}^i_t(dx) \right\}, \ t \in [0,T],$$

and

$$Z_t := \sup_{\alpha_t \in \mathbb{R}^{\mathcal{X}}} \left\{ \langle \alpha_t, \dot{\hat{\mu}}_t - \bar{\Lambda}^*_{\hat{\mu}_t, \hat{m}_t} \hat{\mu}_t \rangle - \int_{\mathcal{X} \times \mathcal{E}_{\mathcal{X}}} \tau(D\alpha_t(x, \Delta)) \bar{\lambda}_{x,x+d\Delta}(\hat{\mu}_t, \hat{m}_t) \hat{\mu}_t(dx) \right\}, \ t \in [0,T].$$

Since $I^*(\hat{\mu}, \hat{\theta}) < +\infty$ it follows that $Z_t < +\infty$ for almost all $t \in [0,T]$. Thanks to the assumption

$\inf_{t\in[\delta,T]}\min_{x\in\mathcal{X}}\hat{\mu}_t(x) > 0$ for all $\delta > 0$, using the Berge's maximum theorem, for almost all $t \in [\eta, T]$, we see that the mapping

$$u \mapsto \sup_{\alpha_t\in\mathbb{R}^{\mathcal{X}}}\left\{\langle \alpha_t, \dot{\hat{\mu}}_t - \bar{\Lambda}^*_{u,\hat{m}_t}u\rangle - \int_{\mathcal{X}\times\mathcal{E}_{\mathcal{X}}} \tau(D\alpha_t(x,\Delta))\bar{\lambda}_{x,x+d\Delta}(u,\hat{m}_t)u(dx)\right\}$$

on $\mathcal{M}_1(\mathcal{X})$ is continuous at $\hat{\mu}_t$. Hence, noting that $Z^i_t = Z_t$ on $t \in [0,\eta]$ for all i sufficiently large, for all $t \in [0,T]\cap\{s \in [0,T] : Z_s < +\infty\}$ we have $\dot{\hat{\mu}}^i_t = \dot{\hat{\mu}}_t$ for all i sufficiently large, and $\hat{\mu}^i_t \to \hat{\mu}_t$ as $i \to \infty$ uniformly in $t \in [0,T]$, it follows that for all $t \in [0,T]\cap\{s \in [0,T] : Z_s < +\infty\}$ $Z^i_t \to Z_t$ as $i \to \infty$. Let us now show the convergence of the corresponding integrals. Fix $t \in (0,T]$ such that $Z_t < +\infty$ and let $\hat{\alpha}^i_t \in \mathbb{R}^{\mathcal{X}}$ and $\hat{\alpha}_t \in \mathbb{R}^{\mathcal{X}}$ attain the supremum in the definition of Z^i_t and Z_t respectively. Whenever $\|\dot{\hat{\mu}}^i_t\| \le i$, we have,

$$\begin{aligned}
0 \le Z^i_t &= \langle \hat{\alpha}^i_t, \dot{\hat{\mu}}^i_t\rangle - \sum_{(x,x')\in\mathcal{E}_{\mathcal{X}}} (\exp\{\hat{\alpha}^i_t(x') - \hat{\alpha}^i_t(x)\} - 1)\bar{\lambda}_{x,x'}(\hat{\mu}^i_t,\hat{m}_t)\hat{\mu}^i_t(x)\\
&= \langle \hat{\alpha}^i_t, \dot{\hat{\mu}}_t\rangle - \sum_{(x,x')\in\mathcal{E}_{\mathcal{X}}} (\exp\{\hat{\alpha}^i_t(x') - \hat{\alpha}^i_t(x)\} - 1)\bar{\lambda}_{x,x'}(\hat{\mu}_t,\hat{m}_t)\hat{\mu}_t(x)\\
&\quad - \sum_{(x,x')\in\mathcal{E}_{\mathcal{X}}} (\exp\{\hat{\alpha}^i_t(x') - \hat{\alpha}^i_t(x)\} - 1) \times (\bar{\lambda}_{x,x'}(\hat{\mu}^i_t,\hat{m}_t)\hat{\mu}^i_t(x) - \bar{\lambda}_{x,x'}(\hat{\mu}_t,\hat{m}_t)\hat{\mu}_t(x))\\
&\le Z_t - \sum_{(x,x')\in\mathcal{E}_{\mathcal{X}}} (\exp\{\hat{\alpha}^i_t(x') - \hat{\alpha}^i_t(x)\} - 1) \times (\bar{\lambda}_{x,x'}(\hat{\mu}^i_t,\hat{m}_t)\hat{\mu}^i_t(x) - \bar{\lambda}_{x,x'}(\hat{\mu}_t,\hat{m}_t)\hat{\mu}_t(x)).
\end{aligned} \tag{3.37}$$

Since $\hat{\mu}^i_t = \hat{\mu}_t$, $t \in [0,\eta]$, for all large enough i, the second term above vanishes whenever $t \in [0,\eta]$. Since $\hat{\mu}^i_t \to \hat{\mu}_t$ as $i \to \infty$ uniformly in $t \in [0,T]$, the first order optimality condition for $(\hat{\alpha}^i_t(x), x \in \mathcal{X})$ (see (3.33)) implies that, for some constant $c_\eta > 0$, we have

$$\max_{(x,x')\in\mathcal{E}_{\mathcal{X}}} \exp\{\hat{\alpha}^i_t(x') - \hat{\alpha}^i_t(x)\} \le c_\eta(1 + \|\dot{\hat{\mu}}_t\|)$$

whenever $t \in [\eta,T]\cap\{s \in [0,T] : Z_s < +\infty\}$. In particular, the right hand side of (3.37) is integrable. Hence, noting that $Z^i_t = 0$ in the alternative case when $\|\dot{\hat{\mu}}_t\| > i$, by an application of the dominated convergence theorem, we have that

$$\int_{[0,T]}\sup_{\alpha_t\in\mathbb{R}^{\mathcal{X}}}\left\{\langle \alpha_t, \dot{\hat{\mu}}^i_t - \bar{\Lambda}^*_{\hat{\mu}^i_t,\hat{m}_t}\hat{\mu}^i_t\rangle - \int_{\mathcal{X}\times\mathcal{E}_{\mathcal{X}}} \tau(D\alpha_t(x,\Delta))\bar{\lambda}_{x,x+d\Delta}(\hat{\mu}^i_t,\hat{m}_t)\hat{\mu}^i_t(dx)\right\}dt$$

converges to

$$\int_{[0,T]} \sup_{\alpha_t \in \mathbb{R}^{\mathcal{X}}} \left\{ \langle \alpha_t, \dot{\hat{\mu}}_t - \bar{\Lambda}^*_{\hat{\mu}_t, \hat{m}_t} \hat{\mu}_t \rangle - \int_{\mathcal{X} \times \mathcal{E}_{\mathcal{X}}} \tau(D\alpha_t(x, \Delta)) \bar{\lambda}_{x, x+d\Delta}(\hat{\mu}_t, \hat{m}_t) \hat{\mu}_t(dx) \right\} dt$$

as $i \to \infty$. Hence, combining the convergences for the slow and the fast components, we have $I^*(\hat{\mu}^i, \hat{\theta}) \to I^*(\hat{\mu}, \hat{\theta})$ as $i \to \infty$. Further, by Remark 3.2, it follows that $\tilde{I}(\hat{\mu}, \hat{\theta}) = I^*(\hat{\mu}, \hat{\theta})$.

In the general case when the mapping $t \mapsto \hat{\mu}_t$ is not locally Lipschitz continuous at $t = 0$, using arguments similar to those used in the proof of Lemma 3.3, one constructs a sequence $\hat{\tau}^i$, $i \geq 1$, and a sequence of elements $(\hat{\mu}^i, \hat{\theta}^i) \in D([0,T], \mathcal{M}_1(\mathcal{X})) \times D_{\uparrow}([0,T], \mathcal{M}(\mathcal{Y}))$, $i \geq 1$, such that $\hat{\tau}^i \to 0$ as $i \to \infty$, $\sup_{t \in [0, \hat{\tau}^i]} \|\dot{\hat{\mu}}^i_t\| < +\infty$ (therefore the mapping $t \mapsto \hat{\mu}^i_t$ is locally Lipschitz continuous at $t = 0$), $(\hat{\mu}^i, \hat{\theta}^i) \to (\hat{\mu}, \hat{\theta})$ in $D([0,T], \mathcal{M}_1(\mathcal{X})) \times D_{\uparrow}([0,T], \mathcal{M}(\mathcal{Y}))$ as $i \to \infty$, $\hat{\mu}^i_t = \hat{\mu}_{t+1/i-\hat{\tau}^i}$ and $\hat{m}^i_t = \hat{m}_{t+1/i-\hat{\tau}^i}$ for all $t \in [\hat{\tau}^i, T]$, and

$$\begin{aligned} \int_{[0,\hat{\tau}^i] \cup [T+1/i-\hat{\tau}^i, T]} \sup_{\alpha_t \in \mathbb{R}^{\mathcal{X}}} & \left\{ \langle \alpha_t, \dot{\hat{\mu}}^i_t - \bar{\Lambda}^*_{\hat{\mu}^i_t, \hat{m}^i_t} \hat{\mu}^i_t \rangle \right. \\ & \left. - \int_{\mathcal{X} \times \mathcal{E}_{\mathcal{X}}} \tau(D\alpha_t(x, \Delta)) \bar{\lambda}_{x, x+d\Delta}(\hat{\mu}^i_t, \hat{m}^i_t) \hat{\mu}^i_t(dx) \right\} dt \\ + \int_{[0,\hat{\tau}^i] \cup [T+1/i-\hat{\tau}^i, T]} \sup_{g_t \in B(\mathcal{Y})} & \left\{ - \int_{\mathcal{Y}} \left(L_{\hat{\mu}^i_t} g_t(\cdot)(y) \right. \right. \\ & \left. \left. + \int_{\mathcal{E}_{\mathcal{Y}}} \tau(Dg_t(y, \Delta)) \gamma_{y, y+d\Delta}(\hat{\mu}^i_t) \right) \hat{m}^i_t(dy) \right\} dt \end{aligned}$$

converges to 0 as $i \to \infty$ (by using the small cost construction of constant velocity paths). Based on what we have already shown for paths that are locally Lipschitz continuous at $t = 0$, we see that $\tilde{I}(\hat{\mu}^i, \hat{\theta}^i) = I^*(\hat{\mu}^i, \hat{\theta}^i)$ for all $i \geq 1$. Again, using arguments similar to those used in the proof of Lemma 3.3, we conclude that $I^*(\hat{\mu}^i, \hat{\theta}^i) \to I^*(\hat{\mu}, \hat{\theta})$ as $i \to \infty$. Once again, by Remark 3.2, we have $\tilde{I}(\hat{\mu}, \hat{\theta}) = I^*(\hat{\mu}, \hat{\theta})$. This completes the proof of the lemma. □

We finally show that $\tilde{I}(\mu, \theta) = I^*(\mu, \theta)$ for all $(\mu, \theta) \in D([0,T], \mathcal{M}_1(\mathcal{X})) \times D_{\uparrow}([0,T], \mathcal{M}(\mathcal{Y}))$, by allowing the path μ to hit the boundary of $\mathcal{M}_1(\mathcal{X})$.

Theorem 3.9. *Let $\nu \in \mathcal{M}_1(\mathcal{X})$ and let $\tilde{I} : D([0,T], \mathcal{M}_1(\mathcal{X})) \times D_{\uparrow}([0,T], \mathcal{M}(\mathcal{Y})) \to [0, +\infty]$ be a subsequential rate function such that $\tilde{I}(\mu, \theta) = +\infty$ unless $\mu_0 = \nu$. Then, for all $(\hat{\mu}, \hat{\theta}) \in D([0,T], \mathcal{M}_1(\mathcal{X})) \times D_{\uparrow}([0,T], \mathcal{M}(\mathcal{Y}))$, we have $\tilde{I}(\hat{\mu}, \hat{\theta}) = I^*(\hat{\mu}, \hat{\theta})$.*

Proof. Since $\tilde{I}(\mu, \theta) \geq I^*(\mu, \theta)$ for all $(\mu, \theta) \in D([0,T], \mathcal{M}_1(\mathcal{X})) \times D_{\uparrow}([0,T], \mathcal{M}(\mathcal{Y}))$, it suffices to focus on a $(\hat{\mu}, \hat{\theta}) \in D([0,T], \mathcal{M}_1(\mathcal{X})) \times D_{\uparrow}([0,T], \mathcal{M}(\mathcal{Y}))$ such that $I^*(\hat{\mu}, \hat{\theta}) < +\infty$ and $\hat{\mu}_0 = \nu$.

By Lemma 3.1, we have that the mapping $[0,T] \ni t \mapsto \hat{\mu}_t \in \mathcal{M}_1(\mathcal{X})$ is absolutely continuous. In particular, $\dot{\hat{\mu}}_t$ exists for almost all $t \in [0,T]$ and $\hat{\mu}_t = \nu + \int_{[0,t]} \dot{\hat{\mu}}_s ds$ for all $t \in [0,T]$.

We shall construct a sequence of paths $\hat{\mu}^i \in D([0,T], \mathcal{M}_1(\mathcal{X}))$, $i \geq 1$, such that $\hat{\mu}^i \to \hat{\mu}$ in $D([0,T], \mathcal{M}_1(\mathcal{X}))$ as $i \to \infty$, $\tilde{I}(\hat{\mu}^i, \hat{\theta}) = I^*(\hat{\mu}, \hat{\theta})$ for all $i \geq 1$, and $I^*(\hat{\mu}^i, \hat{\theta}) \to I^*(\hat{\mu}, \hat{\theta})$ as $i \to \infty$.

Let $\varepsilon_i(x) = \frac{\hat{\mu}_{1/i}(x)+1/i}{1+|\mathcal{X}|/i}$, $x \in \mathcal{X}$ and $i \geq 1$. Using arguments similar to those used in the proof of Lemma 3.3, we first construct a sequence of times $\hat{\tau}^i$, $i \geq 1$, and a sequence of piecewise constant velocity trajectories $\hat{\mu}^i_t$, $t \in [0, \hat{\tau}^i]$, with the property that $\hat{\mu}^i_0 = \hat{\mu}_0$ for all $i \geq 1$, $\hat{\mu}^i_{\hat{\tau}^i}(x) = \varepsilon_i(x)$ for all $x \in \mathcal{X}$ and $i \geq 1$, $\hat{\tau}^i \to 0$ as $i \to \infty$, and

$$\int_{[0,\hat{\tau}^i]} \sup_{\alpha_t \in \mathbb{R}^{\mathcal{X}}} \left\{ \langle \alpha_t, \dot{\hat{\mu}}^i_t - \bar{\Lambda}^*_{\hat{\mu}^i_t, \hat{m}_t} \hat{\mu}^i_t \rangle - \int_{\mathcal{X} \times \mathcal{E}_{\mathcal{X}}} \tau(D\alpha_t(x,\Delta)) \bar{\lambda}_{x,x+d\Delta}(\hat{\mu}^i_t, \hat{m}_t) \hat{\mu}^i_t(dx) \right\} dt \to 0 \quad (3.38)$$

as $i \to \infty$. We then define the path $\hat{\mu}^i_t$ on $t \in (\hat{\tau}^i, T]$ by

$$\hat{\mu}^i_t(x) = \frac{\hat{\mu}_{t+1/i-\hat{\tau}^i}(x) + 1/i}{1 + |\mathcal{X}|/i}, x \in \mathcal{X}.$$

Clearly, $\hat{\mu}^i_t \to \hat{\mu}_t$ as $i \to \infty$ uniformly in $t \in [0,T]$ and hence $\hat{\mu}^i \to \hat{\mu}$ in $D([0,T], \mathcal{M}_1(\mathcal{X}))$ as $i \to \infty$. Note that $(\hat{\mu}^i, \hat{\theta})$ satisfies the conditions of Lemma 3.5 and hence we have $\tilde{I}(\hat{\mu}^i, \hat{\theta}) = I^*(\hat{\mu}^i, \hat{\theta})$ for all $i \geq 1$.

We now show that $I^*(\hat{\mu}^i, \hat{\theta}) \to I^*(\hat{\mu}, \hat{\theta})$ as $i \to \infty$. Using arguments similar to those used in the proof of Lemma 3.5, it is easy to show that

$$\begin{aligned} &\int_{[0,T]} \sup_{g_t \in B(\mathcal{Y})} \left\{ -\int_{\mathcal{Y}} \left(L_{\hat{\mu}^i_t} g_t(\cdot)(y) + \int_{\mathcal{E}_{\mathcal{Y}}} \tau(Dg_t(y,\Delta)) \gamma_{y,y+d\Delta}(\hat{\mu}^i_t) \right) \hat{m}_t(dy) \right\} dt \\ &\quad \to \int_{[0,T]} \sup_{g_t \in B(\mathcal{Y})} \left\{ -\int_{\mathcal{Y}} \left(L_{\hat{\mu}_t} g_t(\cdot)(y) + \int_{\mathcal{E}_{\mathcal{Y}}} \tau(Dg_t(y,\Delta)) \gamma_{y,y+d\Delta}(\hat{\mu}_t) \right) \hat{m}_t(dy) \right\} dt \quad (3.39) \end{aligned}$$

as $i \to \infty$.

To show convergence of the integral corresponding to the slow process, define

$$\begin{aligned} Z^i_t := \sup_{\alpha_t \in \mathbb{R}^{\mathcal{X}}} \Big\{ &\langle \alpha_t, \dot{\hat{\mu}}^i_{t-1/i+\hat{\tau}^i} - \bar{\Lambda}^*_{\hat{\mu}^i_{t-1/i+\hat{\tau}^i}, \hat{m}_t} \hat{\mu}^i_{t-1/i+\hat{\tau}^i} \rangle \\ &- \int_{\mathcal{X} \times \mathcal{E}_{\mathcal{X}}} \tau(D\alpha_t(x,\Delta)) \bar{\lambda}_{x,x+d\Delta}(\hat{\mu}^i_{t-1/i+\hat{\tau}^i}, \hat{m}_t) \hat{\mu}^i_{t-1/i+\hat{\tau}^i}(dx) \Big\}, \end{aligned}$$

$t \in [1/i, T + 1/i - \hat{\tau}^i]$, and

$$Z_t := \sup_{\alpha_t \in \mathbb{R}^{\mathcal{X}}} \Big\{ \langle \alpha_t, \dot{\hat{\mu}}_t - \bar{\Lambda}^*_{\hat{\mu}_t, \hat{m}_t} \hat{\mu}_t \rangle - \int_{\mathcal{X} \times \mathcal{E}_{\mathcal{X}}} \tau(D\alpha_t(x, \Delta)) \bar{\lambda}_{x, x+d\Delta}(\hat{\mu}_t, \hat{m}_t) \hat{\mu}_t(dx) \Big\},\ t \in [0, T].$$

Note the shift in the time index in the definition of Z_t^i to enable direct comparison between Z_t and Z_t^i. For $t \in [1/i, T]$, we then have

$$Z_t^i = \frac{1}{1 + |\mathcal{X}|/i} \sup_{\alpha_t \in \mathbb{R}^{\mathcal{X}}} \Big\{ \langle \alpha_t, \dot{\hat{\mu}}_t \rangle - \sum_{(x,x') \in \mathcal{E}_{\mathcal{X}}} (\exp\{\alpha_t(x') - \alpha_t(x)\} - 1) \bar{\lambda}_{x,x'}(\hat{\mu}^i_{t-1/i+\hat{\tau}^i}, \hat{m}_t)(\hat{\mu}_t(x) + 1/i) \Big\}.$$

The objective function above can be simplified as

$$\begin{aligned}
&\langle \alpha_t, \dot{\hat{\mu}}_t \rangle - \sum_{(x,x') \in \mathcal{E}_{\mathcal{X}}} (\exp\{\alpha_t(x') - \alpha_t(x)\} - 1) \bar{\lambda}_{x,x'}(\hat{\mu}^i_{t-1/i+\hat{\tau}^i}, \hat{m}_t)(\hat{\mu}_t(x) + 1/i) \\
&= \langle \alpha_t, \dot{\hat{\mu}}_t \rangle - \sum_{(x,x') \in \mathcal{E}_{\mathcal{X}}} (\exp\{\alpha_t(x') - \alpha_t(x)\} - 1) \bar{\lambda}_{x,x'}(\hat{\mu}_t, \hat{m}_t) \hat{\mu}_t(x) \\
&\quad - \sum_{(x,x') \in \mathcal{E}_{\mathcal{X}}} (\exp\{\alpha_t(x') - \alpha_t(x)\} - 1) \Big[(\bar{\lambda}_{x,x'}(\hat{\mu}^i_{t-1/i+\hat{\tau}^i}, \hat{m}_t) - \bar{\lambda}_{x,x'}(\hat{\mu}_t, \hat{m}_t)) \hat{\mu}_t(x) \\
&\qquad + \frac{\bar{\lambda}_{x,x'}(\hat{\mu}^i_{t-1/i+\hat{\tau}^i}, \hat{m}_t)}{i} \Big] \\
&\le \langle \alpha_t, \dot{\hat{\mu}}_t \rangle - \sum_{(x,x') \in \mathcal{E}_{\mathcal{X}}} (\exp\{\alpha_t(x') - \alpha_t(x)\} - 1) \bar{\lambda}_{x,x'}(\hat{\mu}_t, \hat{m}_t) \hat{\mu}_t(x) \\
&\quad - \sum_{(x,x') \in \mathcal{E}_{\mathcal{X}}} \exp\{\alpha_t(x') - \alpha_t(x)\} \Big(- \frac{c_L \hat{\mu}_t(x)}{i} + \frac{c}{i} \Big) + |\mathcal{E}_{\mathcal{X}}| \Big(\frac{c_L + C}{i} \Big);
\end{aligned}$$

here $c = \min_{(x,x') \in \mathcal{X}} \min_{y \in \mathcal{Y}} \min_{\xi \in \mathcal{M}_1(\mathcal{X})} \lambda_{x,x'}(\xi, y)$, $C = \max_{(x,x') \in \mathcal{X}} \max_{y \in \mathcal{Y}} \max_{\xi \in \mathcal{M}_1(\mathcal{X})} \lambda_{x,x'}(\xi, y)$, $c_L = \max_{(x,x') \in \mathcal{E}_{\mathcal{X}}} \max_{y \in \mathcal{Y}} c_L^{x,x',y}$ where $c_L^{x,x',y}$ is the Lipschitz constant of $\lambda_{x,x'}(\cdot, y), (x, x') \in \mathcal{E}_{\mathcal{X}}, y \in \mathcal{Y}$, and the last inequality is a consequence of assumption (C2). Fix $t \in [1/i, T + 1/i - \hat{\tau}^i]$ with $Z_t < +\infty$ and let $(\hat{\alpha}_t^i(x), x \in \mathcal{X}) \in \mathbb{R}^{\mathcal{X}}$ denote the optimiser in the definition of Z_t^i. Then the above computation gives us

$$\begin{aligned}
Z_t^i &\leq \frac{1}{1+|\mathcal{X}|/i}\left\{\langle \hat{\alpha}_t^i, \dot{\hat{\mu}}_t\rangle - \sum_{(x,x')\in\mathcal{E}_{\mathcal{X}}}(\exp\{\hat{\alpha}_t^i(x') - \hat{\alpha}_t^i(x)\} - 1)\bar{\lambda}_{x,x'}(\hat{\mu}_t, \hat{m}_t)\hat{\mu}_t(x)\right\} \\
&\quad - \frac{1}{1+|\mathcal{X}|/i}\left\{\sum_{(x,x')\in\mathcal{E}_{\mathcal{X}}}\exp\{\hat{\alpha}_t^i(x') - \hat{\alpha}_t^i(x)\}\left(-\frac{c_L\hat{\mu}_t(x)}{i} + \frac{c}{i}\right) + |\mathcal{E}_{\mathcal{X}}|\left(\frac{c_L + C}{i}\right)\right\} \\
&\leq \frac{1}{1+|\mathcal{X}|/i}Z_t \\
&\quad - \frac{1}{1+|\mathcal{X}|/i}\left\{\sum_{(x,x')\in\mathcal{E}_{\mathcal{X}}}\exp\{\hat{\alpha}_t^i(x') - \hat{\alpha}_t^i(x)\}\left(-\frac{c_L\hat{\mu}_t(x)}{i} + \frac{c}{i}\right) + |\mathcal{E}_{\mathcal{X}}|\left(\frac{c_L + C}{i}\right)\right\}.
\end{aligned}$$

If $\hat{\mu}_t(x) < c/c_L$ for some $x \in \mathcal{X}$, we see that all the terms in the summation corresponding to the edges $(x,x') \in \mathcal{E}_{\mathcal{X}}$ are negative. On the other hand, if $\hat{\mu}_t(x) > c/c_L$, noting that $\hat{\tau}^i \to 0$ as $i \to \infty$ and the convergence of $\hat{\mu}_t^i$ to $\hat{\mu}_t$ as $i \to \infty$ uniformly in $t \in [0,T]$, the first order optimality condition for $(\hat{\alpha}_t^i(x), x \in \mathcal{X})$ implies that, for some constant $c_2 > 0$,

$$\max_{x'\in\mathcal{X}:(x,x')\in\mathcal{E}_{\mathcal{X}}} \exp\{\hat{\alpha}_t^i(x') - \hat{\alpha}_t^i(x)\} \leq c_2(1 + \|\dot{\hat{\mu}}_t\|),$$

and hence for all $t \in [1/i, T + 1/i - \hat{\tau}^i]$ with $Z_t < +\infty$, we obtain that

$$Z_t^i \leq \frac{1}{1+|\mathcal{X}|/i}\{Z_t + c_2|\mathcal{E}_{\mathcal{X}}|(1 + \|\dot{\hat{\mu}}_t\|) + (c_L + C)\}.$$

Hence by the dominated convergence theorem, we see that

$$\begin{aligned}
\int_{[0,T]} \sup_{\alpha_t\in\mathbb{R}^{\mathcal{X}}} \Big\{&\langle \alpha_t, \dot{\hat{\mu}}_t^i - \bar{\Lambda}^*_{\hat{\mu}_t^i, \hat{m}_t}\hat{\mu}_t^i\rangle \\
&- \int_{\mathcal{X}\times\mathcal{E}_{\mathcal{X}}} \tau(D\alpha_t(x,\Delta))\bar{\lambda}_{x,x+d\Delta}(\hat{\mu}_t^i, \hat{m}_t)\hat{\mu}_t^i(dx)\Big\} \times \mathbf{1}_{\{t\geq\hat{\tau}^i\}}dt
\end{aligned}$$

converges to

$$\int_{[0,T]} \sup_{\alpha_t\in\mathbb{R}^{\mathcal{X}}} \left\{\langle \alpha_t, \dot{\hat{\mu}}_t - \bar{\Lambda}^*_{\hat{\mu}_t, \hat{m}_t}\hat{\mu}_t\rangle - \int_{\mathcal{X}\times\mathcal{E}_{\mathcal{X}}} \tau(D\alpha_t(x,\Delta))\bar{\lambda}_{x,x+d\Delta}(\hat{\mu}_t, \hat{m}_t)\hat{\mu}_t(dx)\right\}dt$$

as $i \to \infty$. This along with the convergences (3.38) and (3.39) implies that $I^*(\hat{\mu}^i, \hat{\theta}) \to I^*(\hat{\mu}, \hat{\theta})$ as $i \to \infty$. The procedure of Remark 3.2 then completes the proof of the theorem. □

3.7 Completing the Proof of Theorem 3.1

We finally complete the proof of Theorem 3.1 by extending the conclusion of Theorem 3.9 to all subsequential rate functions $\tilde{I}$, i.e. we remove the restriction that, for some $\nu \in \mathcal{M}_1(\mathcal{X})$, $\tilde{I}(\mu, \theta) = +\infty$ unless $\mu_0 = \nu$.

Proof of Theorem 3.1. Fix $\nu \in \mathcal{M}_1(\mathcal{X})$ and suppose that $\{\mu^N, N \geq 1\}$ is such that $\limsup_{N\to\infty} \frac{1}{N} \log P(|\mu^N(0) - \nu| \geq \varepsilon) = -\infty$ for each $\varepsilon > 0$. By Theorem 3.4, the family $\{(\mu^N, \theta^N), N \geq 1\}$ is exponentially tight in $D([0,T], \mathcal{M}_1(\mathcal{X})) \times D_\uparrow([0,T], \mathcal{M}(\mathcal{Y}))$. Therefore, there exists a subsequence $\{N_k, k \geq 1\}$ of $\mathbb{N}$ such that $\{(\mu^{N_k}, \theta^{N_k}), k \geq 1\}$ satisfies the LDP with rate function $\tilde{I}$ (see, for example, Dembo and Zeitouni [29, Lemma 4.1.23]); by the above condition on the family $\{\mu^N\}$ and by the contraction principle, we see that $\tilde{I}(\mu, \theta) = +\infty$ unless $\mu_0 = \nu$. Therefore, by Theorem 3.9, $\tilde{I} = I^*$ on $D([0,T], \mathcal{M}_1(\mathcal{X})) \times D_\uparrow([0,T], \mathcal{M}(\mathcal{Y}))$. Hence $\tilde{I}$ is uniquely determined for all such subsequences, and it follows that the family $\{(\mu^N, \theta^N), N \geq 1\}$ satisfies the LDP with rate function I^* (see, for example, Dembo and Zeitouni [29, Exercise 4.4.15 (b)]) defined as follows: $I^*(\mu, \theta)$ is defined by (3.7) whenever μ is such that $\mu(0) = \nu$, and $I^*(\mu, \theta) = +\infty$ otherwise.

In the general case when $\{\mu^N(0)\}$ satisfies the LDP on $\mathcal{M}_1(\mathcal{X})$ with rate function I_0, let $p^{(N)}_{\nu^N}$ denote the regular conditional distribution of (μ^N, θ^N) on $D([0,T], \mathcal{M}_1(\mathcal{X})) \times D_\uparrow([0,T], \mathcal{M}(\mathcal{Y}))$ given $\mu^N(0) = \nu^N \in \mathcal{M}_1^N(\mathcal{X})$. By the above argument, whenever $\nu^N \to \nu$ in $\mathcal{M}_1(\mathcal{X})$, $p^{(N)}_{\nu^N}$ satisfies the LDP on $D([0,T], \mathcal{M}_1(\mathcal{X})) \times D_\uparrow([0,T], \mathcal{M}(\mathcal{Y}))$ with rate function $I^*(\mu, \theta) + \infty \mathbf{1}_{\{\mu(0) \neq \nu\}}$. Therefore, the family $\{(\mu^N, \theta^N), N \geq 1\}$ satisfies the LDP on $D([0,T], \mathcal{M}_1(\mathcal{X})) \times D_\uparrow([0,T], \mathcal{M}(\mathcal{Y}))$ with rate function $I_0(\mu(0)) + I^*(\mu, \theta)$ (see, for example, Chaganty [24]). This completes the proof of Theorem 3.1. □

Chapter 4

Large Deviations of the Invariant Measure of Countable-State Mean-Field Models

4.1 The setting and main results

4.1.1 Introduction

For a broad class of Markov processes such as small-noise diffusions, finite-state mean-field models, simple exclusion processes, etc., it is well-known that the Freidlin-Wentzell quasipotential is the rate function that governs the large deviation principle (LDP) for the family of invariant measures [37, 80, 15, 35]. The quasipotential is the minimum cost (arising from the rate function for a process-level large deviation principle) associated with trajectories of arbitrary but finite duration, with fixed initial and terminal conditions. We begin this chapter with two counterexamples of independently evolving countable-state particle systems for which the quasipotential is not the rate function for the family of invariant measures. The family of invariant measures of these counterexamples satisfy the LDP with a suitable relative entropy as its rate function, and we show that the quasipotential is not the same as this relative entropy. Specifically, we show that there are points in the state space where the rate function is finite, but the quasipotential is infinite. These points cannot be reached easily via trajectories of arbitrary but finite time duration. However the barriers to reach these points are surmounted in the stationary regime. There are however some sufficient conditions, at least on a family of such countable-state interacting particle systems, where the Freidlin-Wentzell quasipotential is indeed the correct rate function; this will be the main result of this chapter. Intuitively, the sufficient conditions cut-down the speed of outward excursions and ensure that the insurmountable barriers for the finite horizon trajectories continue to be insurmountable in the stationary regime.

Before we describe the counterexamples and the main result, let us introduce some notations and describe the model of a countable-state mean-field interacting particle system. Let $\mathcal{Z}$ denote the set of non-negative integers and let $(\mathcal{Z}, \mathcal{E})$ denote a directed graph on $\mathcal{Z}$. Let $\mathcal{M}_1(\mathcal{Z})$ denote the space of probability measures on $\mathcal{Z}$ equipped with the total variation metric. For each $N \geq 1$, let $\mathcal{M}_1^N(\mathcal{Z}) \subset \mathcal{M}_1(\mathcal{Z})$ denote the set of probability measures on $\mathcal{Z}$ that can arise as empirical measures of N-particle configurations on $\mathcal{Z}^N$. For each $N \geq 1$, we consider a Markov process with the infinitesimal generator acting on functions f on $\mathcal{M}_1^N(\mathcal{Z})$:

$$\mathscr{L}^N f(\xi) := \sum_{(z,z')\in\mathcal{E}} N\xi(z)\lambda_{z,z'}(\xi)\left[f\left(\xi + \frac{\delta_{z'}}{N} - \frac{\delta_z}{N}\right) - f(\xi)\right], \, \xi \in \mathcal{M}_1^N(\mathcal{Z}); \tag{4.1}$$

here $\lambda_{z,z'} : \mathcal{M}_1(\mathcal{Z}) \to \mathbb{R}_+$, $(z,z') \in \mathcal{E}$, are given functions that describe the transition rates and δ denotes the Dirac measure. Such processes arise as the empirical measure of weakly interacting Markovian mean-field particle systems where the evolution of the state of a particle depends on the states of the other particles only through the empirical measure of the states of all the particles. Under suitable assumptions on the model, the martingale problem for $\mathscr{L}^N$ is well posed and the associated Markov process possesses a unique invariant probability measure $\wp^N$. This chapter highlights certain nuances associated with the large deviation principle for the sequence $\{\wp^N, N \geq 1\}$ on $\mathcal{M}_1(\mathcal{Z})$.

Fix $T > 0$ and let $\mu_{\nu_N}^N$ denote the Markov process with initial condition $\nu_N \in \mathcal{M}_1^N(\mathcal{Z})$ whose infinitesimal generator is $\mathscr{L}^N$. Its sample paths are elements of $D([0,T], \mathcal{M}_1^N(\mathcal{Z}))$, the space of $\mathcal{M}_1^N(\mathcal{Z})$-valued functions on $[0,T]$ that are right-continuous with left limits equipped with the Skorohod topology. Such processes have been well studied in the past. Under mild conditions on the transition rates, when $\nu_N \to \nu$ in $\mathcal{M}_1(\mathcal{Z})$ as $N \to \infty$, it is well-known that the family $\{\mu_{\nu_N}^N, N \geq 1\}$ converges in probability, in $D([0,T], \mathcal{M}_1(\mathcal{Z}))$, as $N \to \infty$ to the *mean-field limit*[1]:

$$\dot{\mu}(t) = \Lambda_{\mu(t)}^* \mu(t), \mu(0) = \nu, \, t \in [0,T]; \tag{4.2}$$

here Λ_ξ, $\xi \in \mathcal{M}_1(\mathcal{Z})$, denotes the rate matrix[2] when the empirical measure is ξ, and $\dot{\mu}(t)$ denotes the derivative of μ at time t. The above dynamical system on $\mathcal{M}_1(\mathcal{Z})$ is called the McKean-

[1] See McKean [59] in the context of interacting diffusions and Bordenave et al. [14] in the context of countable-state mean-field models.

[2] For a $\xi \in \mathcal{M}_1(\mathcal{Z})$, $\Lambda_\xi(z,z') = \lambda_{z,z'}(\xi)$ when $(z,z') \in \mathcal{E}$, $\Lambda_\xi(z,z') = 0$ when $(z,z') \notin \mathcal{E}$, $\Lambda_\xi(z,z) = -\sum_{z'\neq z}\lambda_{z,z'}(\xi)$, and Λ_ξ^* denotes the transpose of Λ_ξ.

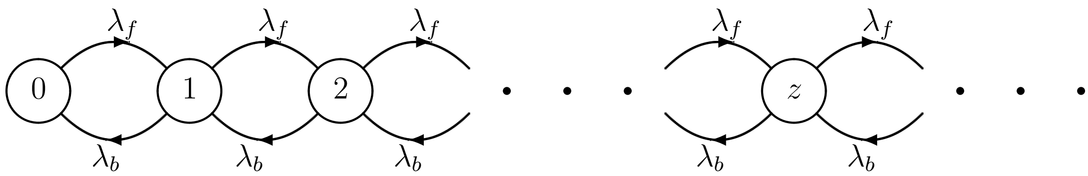

Figure 4.1: Transition rates of an M/M/1 queue.

Vlasov equation. This mean-field convergence allows one to view the process $\mu^N_{\nu_N}$ as a small random perturbation of the dynamical system (4.2). The starting point of our study of the asymptotics of $\{\wp^N, N \geq 1\}$ is the process-level LDP for $\{\mu^N_{\nu_N}, \nu_N \in \mathcal{M}_1^N(\mathcal{Z}), N \geq 1\}$, whenever ν_N converges to ν in $\mathcal{M}_1(\mathcal{Z})$. This LDP was established by Léonard [55] when the initial conditions are fixed, and by Borkar and Sundaresan [15] when the initial conditions converge[1] in $\mathcal{M}_1(\mathcal{Z})$. The rate function of this LDP is governed by "costs" associated with trajectories on $[0,T]$ with initial condition ν, which we denote by $S_{[0,T]}(\varphi|\nu)$, $\varphi \in D([0,T], \mathcal{M}_1(\mathcal{Z}))$ (see (4.5) for its definition).

We assume that ξ^* is the unique globally asymptotically stable equilibrium of (4.2). Define the Freidlin-Wentzell quasipotential

$$V(\xi) \coloneqq \inf\{S_{[0,T]}(\varphi|\xi^*) : \varphi(0) = \xi^*, \varphi(T) = \xi, T > 0\},\ \xi \in \mathcal{M}_1(\mathcal{Z}). \tag{4.3}$$

From the theory of large deviations of the invariant measure of Markov processes [37, 80, 23, 15], V is a natural candidate for the rate function of the family $\{\wp^N, N \geq 1\}$.

4.1.2 Two counterexamples

We begin with two counterexamples for which V is not the rate function for the family of invariant measures.

4.1.2.1 Non-interacting M/M/1 queues

Consider the graph $(\mathcal{Z}, \mathcal{E}_Q)$ whose edge set $\mathcal{E}_Q$ consists of forward edges $\{(z, z+1), z \in \mathcal{Z}\}$ and backward edges $\{(z, z-1), z \in \mathcal{Z} \setminus \{0\}\}$ (see Figure 4.1). Let λ_f and λ_b be two positive

[1]Often, as done in [15], one lets ν_N be random, and only requires $\nu_N \to \nu$ in distribution, where ν is deterministic. For simplicity, we restrict ν_N to be deterministic.

numbers. Consider the generator L^Q acting on functions f on $\mathcal{Z}$ by

$$L^Q f(z) := \sum_{z':(z,z')\in\mathcal{E}_Q} \lambda_{z,z'}(f(z') - f(z)),\ z \in \mathcal{Z},$$

where $\lambda_{z,z+1} = \lambda_f$ for each $z \in \mathcal{Z}$ and $\lambda_{z,z-1} = \lambda_b$ for each $z \in \mathcal{Z} \setminus \{0\}$. When $\lambda_f < \lambda_b$, the invariant probability measure associated with this Markov process is

$$\xi_Q^*(z) := \left(1 - \frac{\lambda_f}{\lambda_b}\right)\left(\frac{\lambda_f}{\lambda_b}\right)^z,\ z \in \mathcal{Z}.$$

For each $N \geq 1$, we consider N particles, each of which evolves independently as a Markov process on $\mathcal{Z}$ with the infinitesimal generator L^Q. That is, the particles are independent M/M/1 queues. It is easy to check that the empirical measure of the system of particles is also a Markov process on the state space $\mathcal{M}_1^N(\mathcal{Z})$ and it possesses a unique invariant probability measure, which we denote by $\wp_Q^N$.

On one hand, it is straightforward to see that the family $\{\wp_Q^N, N \geq 1\}$ satisfies the LDP on $\mathcal{M}_1(\mathcal{Z})$. Indeed, under stationarity, the state of each particle is distributed as ξ_Q^*. As a consequence, $\wp_Q^N$ is the law of the random variable $\frac{1}{N}\sum_{n=1}^N \delta_{\zeta_n}$ on $\mathcal{M}_1(\mathcal{Z})$, where $\zeta_1, \ldots, \zeta_N$ are independent and identically distributed (i.i.d.) as ξ_Q^*. Therefore, by Sanov's theorem [29, Theorem 6.2.10], $\{\wp_Q^N, N \geq 1\}$ satisfies the LDP with the rate function $I(\cdot\|\xi_Q^*)$, where $I : \mathcal{M}_1(\mathcal{Z}) \times \mathcal{M}_1(\mathcal{Z}) \to [0,\infty]$ is the relative entropy defined by[1]

$$I(\zeta\|\nu) := \begin{cases} \sum_{z\in\mathcal{Z}} \zeta(z) \log\left(\frac{\zeta(z)}{\nu(z)}\right), & \text{if } \zeta \ll \nu, \\ \infty, & \text{otherwise.} \end{cases} \tag{4.4}$$

On the other hand, it is natural to conjecture that the rate function for the family $\{\wp_Q^N, N \geq 1\}$ is given by the quasipotential (4.3) with ξ^* replaced by ξ_Q^*. However, as discussed in the next paragraph, the quasipotential is not the same as $I(\cdot\|\xi_Q^*)$. Hence, from the uniqueness of the large deviations rate function [29, Lemma 4.1.4], the quasipotential does not govern the rate function for the family $\{\wp_Q^N, N \geq 1\}$.

We now provide some intuition on why the quasipotential is not the rate function in the example under consideration. For a formal proof, see Section 4.8. Let $\vartheta(z) = z \log z$, and let $\iota(z) = z$, $z \in \mathcal{Z}$. Using the fact that ξ_Q^* has geometric decay, it can be checked that $I(\xi\|\xi_Q^*)$ is finite if and only if the first moment of ξ (denoted by $\langle \xi, \iota\rangle$) is finite. However it turns out that

[1] We use the convention $0 \log 0 = 0$.

$V(\xi)$ (i.e., the quantity in (4.3) with ξ^* replaced by ξ_Q^*) is finite if and only if the ϑ-moment of ξ (denoted by $\langle \xi, \vartheta \rangle$) is finite. In particular, if we consider a $\xi \in \mathcal{M}_1(\mathcal{Z})$ whose first moment is finite but ϑ-moment is infinite then $V(\xi) \neq I(\xi \| \xi_Q^*)$. Let $\varepsilon > 0$, $\xi \in \mathcal{M}_1(\mathcal{Z})$ be such that $\langle \xi, \iota \rangle < \infty$ but $\langle \xi, \vartheta \rangle = \infty$, and consider the ε-neighbourhood of ξ in $\mathcal{M}_1(\mathcal{Z})$. By Sanov's theorem, the probability of this neighbourhood under $\wp_Q^N$ is of the form $\exp\{-N(I(\xi \| \xi_Q^*) + o(1))\}$. For a fixed $T > 0$, let us now try to estimate the probability of $\mu_{\nu_N}^N(T)$ being in this neighbourhood when ν_N is in a small neighbourhood of ξ_Q^*. If the process μ^N is initiated at a ν_N near ξ_Q^*, then the probability that the random variable $\mu_{\nu_N}^N(T)$ is in the ε-neighbourhood of ξ is at most

$$\exp\left\{-N\left(\inf_{\{\xi':\mathrm{dist}(\xi,\xi')\leq\varepsilon\}} V(\xi') + o(1)\right)\right\}.$$

Since V is lower semicontinuous (we prove this in Lemma 4.7), we must have

$$\inf_{\{\xi':\mathrm{dist}(\xi,\xi')\leq\varepsilon\}} V(\xi') \to \infty \text{ as } \varepsilon \to 0.$$

Hence we can choose an ε small enough so that $\inf_{\{\xi':\mathrm{dist}(\xi,\xi')\leq\varepsilon\}} V(\xi') > 2I(\xi \| \xi_Q^*)$. For this ε, the probability that $\mu_{\nu_N}^N(T)$ lies is the ε-neighbourhood of ξ is upper bounded by $\exp\{-N \times (2I(\xi \| \xi_Q^*) + o(1))\}$, which is smaller than $\exp\{-N(I(\xi \| \xi_Q^*) + o(1))\}$, even in the exponential scale, for large enough N. That is, for any arbitrary but fixed T, we can find a small neighbourhood of ξ such that the probability that $\mu_{\nu_N}^N(T)$ lies in that neighbourhood is smaller than what we expect to see in the stationary regime. In other words, there are some barriers in $\mathcal{M}_1(\mathcal{Z})$ that cannot be surmounted in any finite time, yet these barriers can be crossed in the stationary regime. These barriers indicate that, to obtain the correct stationary regime probability of a small neighbourhood of ξ using the dynamics of $\mu_{\nu_N}^N$, one should wait longer than any fixed time horizon. That is, one should consider the random variable $\mu_{\nu_N}^N(T(N))$, where $T(N)$ is a suitable function of N, and estimate the probability that $\mu_{\nu_N}^N(T(N))$ belongs to a small neighbourhood of ξ. However it is not straightforward to obtain such estimates from the process-level large deviation estimates of $\mu_{\nu_N}^N$ since the latter are usually available for a fixed time duration.

There are natural barriers in the context of finite-state mean-field models when the limiting dynamical system has multiple (but finitely many) stable equilibria (see Section 2.3 in Chapter 2). In such situations, passages from a neighbourhood of one equilibrium to a neighbourhood of another take place over time durations of the form $\exp\{N \times O(1)\}$ where N is the number of particles[1]. Interestingly, these barriers can be surmounted using trajectories of finite

[1] $O(1)$ refers to a bounded sequence, and $\omega(1)$ refers to a sequence that goes to ∞.

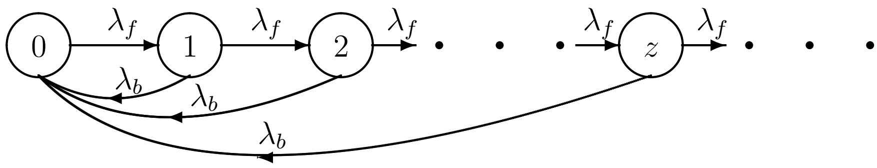

Figure 4.2: Transition rates of a wireless node.

time durations; i.e., for any fixed T, the probability that the empirical measure process reaches a neighbourhood of an equilibrium at time T when it is initiated in a small neighbourhood of another equilibrium is of the form $\exp\{-N \times O(1)\}$. In contrast, in the case of the above counterexample, the barriers cannot be surmounted in finite time durations; for any fixed T, the probability that $\mu^N(T)$ reaches a small neighbourhood of a point in $\mathcal{M}_1(\mathcal{Z})$ with finite first moment but infinite ϑ-moment when it is initiated from a neighbourhood of ξ_Q^* is of the form $\exp\{-N \times \omega(1)\}$. Hence we anticipate that the barriers that we encounter in the above counterexample are somehow more difficult to surmount than those that arise in the case of finite-state mean-field models with multiple stable equilibria.

4.1.2.2 Non-interacting nodes in a wireless network

We provide another counterexample where the issue is similar. Consider the graph $(\mathcal{Z}, \mathcal{E}_W)$ whose edge set $\mathcal{E}_W$ consists of forward edges $\{(z, z+1), z \in \mathcal{Z}\}$ and backward edges $\{(z, 0), z \in \mathcal{Z} \setminus \{0\}\}$ (see Figure 4.2). Let λ_f and λ_b be positive numbers. Consider the generator L^W acting on functions f on $\mathcal{Z}$ by

$$L^W f(z) := \sum_{z':(z,z') \in \mathcal{E}_W} \lambda_{z,z'}(f(z') - f(z)), z \in \mathcal{Z},$$

where $\lambda_{z,z+1} = \lambda_f$ for each $z \in \mathcal{Z}$ and $\lambda_{z,0} = \lambda_b$ for each $z \in \mathcal{Z} \setminus \{0\}$. The invariant probability measure associated with this Markov process is

$$\xi_W^*(z) := \frac{\lambda_b}{\lambda_f + \lambda_b} \left(\frac{\lambda_f}{\lambda_f + \lambda_b} \right)^z, z \in \mathcal{Z}.$$

Similar to the previous example, for each $N \geq 1$, we consider N particles, each of which evolves independently as a Markov process on $\mathcal{Z}$ with the infinitesimal generator L^W. It is easy to check that the empirical measure of the system of particles possesses a unique invariant probability measure, which we denote by $\wp_W^N$. Under stationarity, the state of each particle is distributed as ξ_W^*. As a consequence, $\wp_W^N$ is the law of the random variable $\frac{1}{N}\sum_{n=1}^N \delta_{\zeta_n}$ on $\mathcal{M}_1(\mathcal{Z})$, where $\zeta_1, \ldots, \zeta_N$ are i.i.d. ξ_W^*. Hence, by Sanov's theorem, the family $\{\wp_W^N, N \geq 1\}$

satisfies the LDP with the rate function $I(\cdot\|\xi^*_W)$. As we show in Section 4.8, in this example too, the quasipotential (4.3) with ξ^* replaced by ξ^*_W is not the same as $I(\cdot\|\xi^*_W)$. As in the previous example, there are points ξ where $V(\xi)=\infty$ but $I(\xi\|\xi^*_Q)<\infty$, points ξ that have a finite first moment but infinite ϑ-moment. Once again, the quasipotential does not govern the rate function for the family $\{\wp^N_W, N\geq 1\}$.

4.1.3 Assumptions and main result

We now provide some assumptions on the model of countable-state mean-field interacting particle systems that ensure that the barriers in $\mathcal{M}_1(\mathcal{Z})$ that are insurmountable using trajectories of arbitrary but finite time duration remain insurmountable in the stationary regime as well. Under these assumptions, we prove the main result of this chapter, i.e., the sequence of invariant measures $\{\wp^N, N\geq 1\}$ satisfies the LDP with rate function V.

4.1.3.1 Assumptions

Our first set of assumptions is on the mean-field interacting particle system (i.e., on the generator $\mathscr{L}^N$ defined in (4.1)).

(E1) The edge set is given by $\mathcal{E}=\{(z,z+1), z\in\mathcal{Z}\}\cup\{(z,0), z\in\mathcal{Z}\setminus\{0\}\}$.

(E2) There exist positive constants $\overline{\lambda}$ and $\underline{\lambda}$ such that

$$\frac{\underline{\lambda}}{z+1}\leq\lambda_{z,z+1}(\xi)\leq\frac{\overline{\lambda}}{z+1}, \text{ and } \underline{\lambda}\leq\lambda_{z,0}(\xi)\leq\overline{\lambda},$$

for each $\xi\in\mathcal{M}_1(\mathcal{Z})$.

(E3) The functions $(z+1)\lambda_{z,z+1}(\cdot)$, $z\in\mathcal{Z}$, and $\lambda_{z,0}(\cdot)$, $z\in\mathcal{Z}\setminus\{0\}$, are uniformly Lipschitz continuous on $\mathcal{M}_1(\mathcal{Z})$.

Note that assumption (E1) considers a specific transition graph (Figure 4.2) for each particle. This graph arises in the contexts of random backoff algorithms for medium access in wireless local area networks [51] and decentralised control of loads in a smart grid [60]. Assumption (E2) ensures that the forward transition rates at state z decays as $1/z$. This key assumption cuts down the speed of outward excursions and enables us to overcome the issue described in the counterexamples. To highlight this, consider a modified example of Section 4.1.2.2 where $\lambda_{z,z+1}=\lambda_f/(z+1)$, $z\in\mathcal{Z}$; the rest of the description remains the same. Let $\tilde{\xi}_W\in\mathcal{M}_1(\mathcal{Z})$ denote the invariant probability measure associated with one particle. It can be checked that

$\tilde{\xi}_W(z)$ is of the order of $\exp\{-\vartheta(z)\}$, unlike ξ^*_W which has geometric decay. As a consequence, $I(\xi\|\tilde{\xi}_W)$ is finite if and only if the ϑ-moment of ξ is finite. Hence, by imposing (E2), we have ensured that the barriers in $\mathcal{M}_1(\mathcal{Z})$ that are insurmountable for finite time duration trajectories continue to remain insurmountable in the stationary regime; this is the key property that enables us to prove the main result of this chapter. Assumption (E3) is a uniform Lipschitz continuity property for the transition rates which is required for the process-level LDP for $\mu^N_{\nu_N}$ to hold and for the McKean-Vlasov equation (4.2) to be well-posed.

Our second set of assumptions is on the McKean-Vlasov equation (4.2). Let μ_ν, $\nu \in \mathcal{M}_1(\mathcal{Z})$, denote the solution to the limiting dynamics (4.2) with initial condition $\nu \in \mathcal{M}_1(\mathcal{Z})$. Recall the function ϑ. Define $\mathscr{K}_M \coloneqq \{\xi \in \mathcal{M}_1(\mathcal{Z}) : \langle \xi, \vartheta\rangle \leq M\}$, $M > 0$.

(F1) There exists a unique globally asymptotically stable equilibrium ξ^* for the McKean-Vlasov equation (4.2).

(F2) $\langle \xi^*, \vartheta\rangle < \infty$ and $\lim_{t\to\infty} \sup_{\nu\in\mathscr{K}_M} \langle \mu_\nu(t), \vartheta\rangle = \langle \xi^*, \vartheta\rangle$ for each $M > 0$.

The first assumption above asserts that all the trajectories of (4.2) converge to ξ^* as time becomes large. The proof of the LDP upper and lower bounds for the family $\{\wp^N, N \geq 1\}$ involves construction of trajectories that start at suitable compact sets, reach the stable equilibrium ξ^* using arbitrarily small cost, and then terminate at a desired point in $\mathcal{M}_1(\mathcal{Z})$ starting from ξ^*. All these are enabled by assumption (F1) (see more remarks about this assumption in Section 4.1.4). The second assumption asserts that the ϑ-moment of the solution to the limiting dynamics converges uniformly over initial conditions lying in sets of bounded ϑ-moment. In the case of a non-interacting system that satisfies (E1) but with constant forward transition rates (for example, see L^W in Section 4.1.2.2), the analogue of this assumption can easily be verified: the first moment of the solution to the limiting dynamics converges uniformly over initial conditions lying in sets of bounded first moment. In fact, one can explicitly write down the first moment of the solution to the limiting dynamics in this case and verify this assumption easily. Assumption (F2) is the analogous statement for our mean-field system that satisfies the $1/z$-decay of the forward transition rates in assumption (E2).

4.1.3.2 Main result

We now state the main result of this chapter, namely the LDP for the family of invariant measures $\{\wp^N, N \geq 1\}$ under the assumptions (E1)–(E3) and (F1)–(F2).

We first assert the existence and uniqueness of the invariant measure $\wp^N$ for $\mathscr{L}^N$ for each $N \geq 1$, and the exponential tightness of the family $\{\wp^N, N \geq 1\}$.

Proposition 4.1. *Assume (E1) and (E2). For each $N \geq 1$, $\mathscr{L}^N$ admits a unique invariant probability measure $\wp^N$. Further, the family $\{\wp^N, N \geq 1\}$ is exponentially tight in $\mathcal{M}_1(\mathcal{Z})$.*

Recall the quasipotential V defined in (4.3). We now state the main result of this chapter.

Theorem 4.1. *Assume (E1), (E2), (E3), (F1), and (F2). Then the family of probability measures $\{\wp^N, N \geq 1\}$ satisfies the large deviation principle on $\mathcal{M}_1(\mathcal{Z})$ with rate function V.*

The proof of this result is carried out in Sections 4.4–4.7. We begin with the process-level uniform LDP for $\mu^N_{\nu_N}$ over compact subsets of $\mathcal{M}_1(\mathcal{Z})$; this uniform LDP gives us the large deviation estimates for the process $\mu^N_{\nu_N}$ uniformly over the initial conditions ν_N lying in a given compact set (see Definition 4.2 and Theorem 4.2). We prove the LDP for the family $\{\wp^N, N \geq 1\}$ by transferring this process-level uniform LDP for $\mu^N_{\nu_N}$ over compact subsets of $\mathcal{M}_1(\mathcal{Z})$ to the stationary regime. The proof of the LDP lower bound (in Section 4.4) considers specific trajectories and lower bounds the probability of small neighbourhoods of points in $\mathcal{M}_1(\mathcal{Z})$ under $\wp^N$ using the probability that the process $\mu^N_{\nu_N}$ remains close to these trajectories. For the proof of the upper bound, we require certain regularity properties of the quasipotential. These properties are established in Section 4.5. We first show a controllability[1] property for V: $V(\xi)$ is finite if and only if $\langle \xi, \vartheta \rangle < \infty$. Using the lower bound proved in Section 4.4, we then show that the level sets of V are compact subsets of $\mathcal{M}_1(\mathcal{Z})$. Since $\mathcal{M}_1(\mathcal{Z})$ is not locally compact, this compactness-of-level-sets property implies that there are points in $\mathcal{M}_1(\mathcal{Z})$ where V is discontinuous. However we show the following *small cost connection* property: whenever $\xi_n \to \xi^*$ in $\mathcal{M}_1(\mathcal{Z})$ and $\langle \xi_n, \vartheta \rangle \to \langle \xi^*, \vartheta \rangle$ as $n \to \infty$, we have $\lim_{n\to\infty} V(\xi_n) = V(\xi^*) = 0$. These properties of the quasipotential are then used to transfer the process-level uniform LDP upper bound for $\mu^N_{\nu_N}$ (uniform over compact subsets of $\mathcal{M}_1(\mathcal{Z})$) to the LDP upper bound for the family of invariant measures. The proof of the upper bound is carried out in Section 4.6. Finally, we complete the proof of the theorem in Section 4.7.

While the proofs of our lower and upper bounds follow the general methodology of Sowers [80], there are significant model-specific difficulties that arise in our context. The main novelty in the proof of Theorem 4.1 is to establish the small cost connection property of the quasipotential V under assumptions (E1)–(E3) and (F1)–(F2). That is, we can find trajectories of small cost that start at ξ^* and end at points in $\mathcal{M}_1(\mathcal{Z})$ whose ϑ-moment is not very far from that of ξ^*. In the work of Sowers [80], this has been carried out by considering the "straight-line" trajectory that connects the attractor to the nearby point under consideration. Such a trajectory may not have small cost in our case since the mass transfer is restricted to

[1] This terminology is from Cerrai and Röckner [23].

the edges in $\mathcal{E}$. We overcome this difficulty by considering a piecewise constant velocity mass transfer via the edges in $\mathcal{E}$. We then carefully estimate the cost of this trajectory and prove the necessary *small cost connection* property. We also simplify the proof of the compactness of the lower level sets of V; while Sowers [81, Proposition 7] studies the minimisation of the costs of trajectories over the infinite-horizon, we arrive at it by using the LDP lower bound and the exponential tightness of the family $\{\wp^N, N \geq 1\}$. We also remark that the methodology of Sowers [80] has been used by Cerrai and Röckner [23] in the context of stochastic reaction diffusion equations and by Cerrai and Paskal [21] in the context of two-dimensional stochastic Navier-Stokes equations.

4.1.4 Discussion and future directions

The main result and the counterexamples suggest that in order for the family of invariant measures of a Markov process to satisfy the large derivation principle with rate function governed by the Freidlin-Wentzell quasipotential, one must have some *good* properties on the model under consideration. In the case of our main result, this goodness property was achieved by the $1/z$-decay of the forward transition rates from assumption (E2). We use this assumption to show the exponential tightness of the invariant measure over compact subsets with bounded ϑ-moments. It also enables us to show the necessary regularity properties of the quasipotential required to transfer the process-level large deviation result to the stationary regime. However a general treatment of the LDP for the family of invariant measures of Markov processes (that encompasses the cases of [80, 23, 21, 15, 35]), especially when the ambient state space is not locally compact, is missing in the literature.

One of the assumptions that plays a significant role in the proof of our main result is the existence of a unique globally asymptotically stable equilibrium for the limiting dynamics (assumption (F1))[1]. In general, the limiting dynamical system (4.2) could possess multiple ω-limit sets. In that case the approach of our proofs breaks down. A well-known approach to study large deviations of the invariant measures in such cases is to focus on small neighbourhoods of these ω-limit sets and then analyse the discrete time Markov chain that evolves on these neighbourhoods. The LDP then follows from the estimates of the invariant measure of this discrete time chain (see Freidlin and Wentzell [37, Chapter 6, Section 4]). However this approach requires the *uniform LDP over open subsets* of $\mathcal{M}_1(\mathcal{Z})$, which is not yet available for our mean-field model. If this can be established, along with the regularity properties of the quasipotential

[1] In the works of Sowers [80], Cerrai and Röckner [23], and Cerrai and Paskal [21], their model assumptions ensure that (F1) holds.

established in Section 4.5, one can not only use the above idea to extend our main result to the case when the limiting dynamical system possesses multiple ω-limit sets but also to study exit problems and metastability phenomena in our mean-field model.

Another definition of the quasipotential appears in the literature. It is given by the minimisation of costs of the form $S_{(-\infty,0]}(\varphi)$ over infinite-horizon trajectories φ on $(-\infty, 0]$ such that the terminal time condition $\varphi(0)$ is fixed and $\varphi(t) \to \xi^*$ as $t \to -\infty$ (see Sowers [80], Cerrai and Röckner [23]). While it is clear that the above definition of the quasipotential is a lower bound for V in (4.3), unlike in Sowers [80] and Cerrai and Röckner [23], we are not able to show that the two definitions are the same. A proof of this equality, or otherwise, will add more insight on the general case.

We remark that assumption (E3) does not play a role in the proof of our main result. It is used to invoke the process-level LDP for $\mu^N_{\nu_N}$ (see Theorem 4.2) and the well-posedness of the limiting dynamical system (4.2). If these two properties are established through some other means then the proof of Theorem 4.1 holds verbatim without the need for assumption (E3).

Finally, we mention that a time-independent variational formula for the quasipotential is available for some non-reversible models in statistical mechanics, see Bertini et al. [8, 9]. It is not clear if the quasipotential V in (4.3) admits a time-independent variational form. This would be an interesting direction to explore.

4.1.5 Related literature

Process-level large deviations of small-noise diffusion processes have been well studied in the past. For finite-dimensional large deviation problems, see Freidlin and Wentzell [37, Chapter 5], Liptser [57], Veretennikov [88], Puhalskii [73], and the references therein. For infinite-dimensional problems where the state space is not locally compact, see Sowers [81] and Cerrai and Röckner [22]. More recently, uniform large deviation principle (uniform LDP) for Banach-space valued stochastic differential equations over the class of bounded and open subsets of the Banach space have been studied by Salins et al. [79]. These have been used to study the exit times and metastability in such processes, see Salins and Spiliopoulos [78]. While the above works focus on diffusion processes, our work focuses on the stationary regime large deviations of countable-state mean-field models with jumps. In the spirit of the small-noise problems listed above, our process $\mu^N_{\nu_N}$ can be viewed as a small random perturbation of the dynamical system (4.2) on $\mathcal{M}_1(\mathcal{Z})$.

In the context of interacting particle systems, Dawson and Gärtner [26] established the process-level LDP for weakly interacting diffusion processes, and Léonard [55] and Borkar and

Sundaresan [15] extended this to mean-field interacting particle systems with jumps. In this work, we focus on the stationary regime large deviations of mean-field models with jumps when the state of each particle comes from a countable set. For small-noise diffusion process on Euclidean spaces and finite-state mean-field models, since the state space (on which the empirical measure process evolves) is locally compact, the process-level large deviation results have been extended in a straightforward manner to the uniform LDP over the class of open subsets of the space. Such uniform large deviation estimates have been used to prove the large deviations of the invariant measure and the exit time estimates, see Freidlin and Wentzell [37, Chapter 6] in the context of diffusion processes, Borkar and Sundaresan [15] and Chapter 2 of this thesis in the context of finite-state mean-field models. One of the key ingredients in these proofs is the continuity of the quasipotential. However in our case, the state space $\mathcal{M}_1(\mathcal{Z})$ is infinite-dimensional and not locally compact. Therefore, since the quasipotential (4.3) is expected to have compact lower level sets, it cannot be continuous on $\mathcal{M}_1(\mathcal{Z})$ unlike in the finite-dimensional problems mentioned above. Hence the ideas presented in [15] are not directly applicable to our context of the LDP for the family of invariant measures.

Large deviations of the family of invariant measures for small-noise diffusion processes on non-locally compact spaces have also been studied in the past, see Sowers [80] and Cerrai and Röckner [23]. They have a unique attractor for the limiting dynamics, and the proof essentially involves conversion of the uniform LDP over the finite-time horizon to the stationary regime. Martirosyan [58] studied a situation where the limiting dynamical system possesses multiple attractors. For the study of large deviations of the family of invariant measures for simple exclusion processes, see Bodineau and Giacomin [13] and Bertini et al. [9]. More recently, Farfán et al. [35] extended this to a simple exclusion process whose limiting hydrodynamic equation has multiple attractors. Their proof proceeds similar to the case of finite-dimensional diffusions in Freidlin and Wentzell [37, Chapter 6, Section 4] by first approximating the process near the attractors and then using the Khasminskii reconstruction formula [48, Chapter 4, Section 4]. In particular, it requires the uniform LDP to hold over open subsets of the state space. Since their state space, although infinite-dimensional, is compact, the proof of the uniform LDP over open subsets easily follows from the process-level LDP. Also, the compactness of the state space simplifies the proofs of the *small cost connection* property from the attractors to nearby points, a property needed in the Khasminskii reconstruction. Although we restrict our attention to the case of a unique globally asymptotically stable equilibrium as in [80, 23], the main novelty of our work is that we establish certain regularity properties of the quasipotential for countable-state mean-field models with jumps which were not done in the past. We then use these properties to prove the LDP for the family of invariant measures. Furthermore, we demonstrate two counterexamples where the stationary regime LDP's rate functions are not governed by the usual quasipotential. To the best of our knowledge, such examples where the LDP for the family of invariant measures hold but there rate functions are not governed by the usual Freidlin-Wentzell quasipotential are new. These examples are constructed in a way that the particle systems do not possess the *small cost connection* property from the attractor to nearby points with finite first moment but infinite ϑ-moment. On a related note, counterexamples are known in the literature for small-noise Markov processes where the asymptotics of the spectral gap differs from the natural candidate, which is a quantity analogous Λ defined in (2.9), see Miclo [63].

Large deviations of the family of invariant measure for a queueing network in a finite-dimensional setting has been studied by Puhalskii [72]. Finally, large deviations of the family of invariant measures for a stochastic process under some general conditions has been studied by Puhalskii [74]. One of their conditions is the small cost connection property between any two nearby points in the state space, which cannot be satisfied by our countable-state mean-field model since our state space is not locally compact.

4.1.6 Organisation

This chapter is organised as follows. In Section 4.2, we provide preliminary results on the large deviations over finite time horizons. The proof of the main result is carried out in Sections 4.3–4.7. In Section 4.3, we prove the existence, uniqueness, and exponential tightness of the family of invariant measures. In Section 4.4, we prove the LDP lower bound for the family of invariant measures. In Section 4.5, we establish some regularity properties of the quasipotential V defined in (4.3). In Section 4.6, we prove the LDP upper bound for the family of invariant measures. In Section 4.7, we complete the proof of the main result. Finally in Section 4.8, we prove that the quasipotential differs from the relative entropy (with respect to the globally asymptotically stable equilibrium) for the two counterexamples discussed in Section 4.1.2.

4.2 Preliminaries

4.2.1 Frequently used notation

We first summarise the frequently used notation in the chapter. Let $\mathcal{Z}$ denote the set of nonnegative integers and let $(\mathcal{Z}, \mathcal{E})$ denote a directed graph on $\mathcal{Z}$. Let $\mathbb{R}^\infty$ denote the infinite product of $\mathbb{R}$ equipped with the topology of pointwise convergence (it is also viewed as the space of real-valued functions on $\mathcal{Z}$). Recall that $\mathcal{M}_1(\mathcal{Z})$ denotes the space of probability measure on $\mathcal{Z}$ equipped with the total variation metric (denoted by d). This metric generates the topology of weak convergence on $\mathcal{M}_1(\mathcal{Z})$. By Scheffé's lemma [32, Chapter 3, Section 2], $\mathcal{M}_1(\mathcal{Z})$ can be identified with the subset $\{x \in \mathbb{R}^\infty : x_i \geq 0 \, \forall i, \sum_{i\geq 0} x_i = 1\}$ of $\mathbb{R}^\infty$ with the subspace topology. For each $N \geq 1$, recall that $\mathcal{M}_1^N(\mathcal{Z}) \subset \mathcal{M}_1(\mathcal{Z})$ denotes the space of probability measures on $\mathcal{Z}$ that can arise as empirical measures of N-particle configurations on $\mathcal{Z}^N$. Recall that $\vartheta(z) = z \log z, z \in \mathcal{Z}$, with the convention that $0 \log 0 = 0$. Given $f, g \in \mathbb{R}^\infty$, let the bracket $\langle f, g\rangle$ denote $\lim_{n\to\infty} \sum_{k=0}^n f(k)g(k)$, whenever the limit exists. For $M > 0$, define $\mathscr{K}_M := \{\xi \in \mathcal{M}_1(\mathcal{Z}) : \langle \xi, \vartheta\rangle \leq M\}$; by Prohorov's theorem, $\mathscr{K}_M$ is a compact subset of $\mathcal{M}_1(\mathcal{Z})$. Define $\mathscr{K} := \bigcup_{M\geq 1} \mathscr{K}_M$. Let $\xi^* \in \mathcal{M}_1(\mathcal{Z})$ denote the globally asymptotically stable equilibrium for the McKean-Vlasov equation (4.2) (see assumption (F1)). For each $\Delta > 0$, define

$$K(\Delta) := \{\xi \in \mathscr{K} : d(\xi^*, \xi) \leq \Delta \text{ and } |\langle \xi^*, \vartheta\rangle - \langle \xi, \vartheta\rangle| \leq \Delta\};$$

note that $K(\Delta)$ depends on ξ^* as well (which we do not indicate for ease of readability). Recall the functions τ and τ^* defined in Section 2.2, i.e., $\tau(u) := e^u - u - 1$, $u \in \mathbb{R}$, and

$$\tau^*(u) := \begin{cases} \infty & \text{if } u < -1, \\ 1 & \text{if } u = -1, \\ (u+1)\log(u+1) - u & \text{if } u > -1. \end{cases}$$

For a complete and separable metric space $(\mathcal{S}, d_0)$, $A \subset \mathcal{S}$, and $x \in \mathcal{S}$, let $d_0(x, A)$ denote $\inf_{y\in A} d_0(x, y)$. Let $D([0, T], \mathcal{S})$ denote the space of $\mathcal{S}$-valued functions on $[0, T]$ that are right continuous with left limits. It is equipped with the Skorohod topology which makes it a complete and separable metric space (see, for example, Ethier and Kurtz [34, Chapter 3]). Let ρ denote a metric on $D([0, T], \mathcal{S})$ that generates the Skorohod topology. An element of $D([0, T], \mathcal{S})$ is called a "trajectory", and we shall refer to the process-level large deviations rate function evaluated on a trajectory as the "cost" associated with that trajectory. For a trajectory φ, let both φ_t and $\varphi(t)$ denote the evaluation of φ at time t. For $N \geq 1$ and $\nu \in \mathcal{M}_1^N(\mathcal{Z})$, let $\mathbb{P}_\nu^N$ denote the solution to the $D([0, T], \mathcal{M}_1^N(\mathcal{Z}))$-valued martingale problem for $\mathscr{L}^N$ with initial condition ν (whenever the martingale problem for $\mathscr{L}^N$ is well-posed). Let μ_ν^N denote the random element of $D([0, T], \mathcal{M}_1^N(\mathcal{Z}))$ whose law is $\mathbb{P}_\nu^N$. For each $\xi \in \mathcal{M}_1(\mathcal{Z})$, let L_ξ denote the generator acting on functions f on $\mathcal{Z}$ by

$$f \mapsto L_\xi(z) := \sum_{z':(z,z')\in\mathcal{E}} \lambda_{z,z'}(\xi)(f(z') - f(z)), \; z \in \mathcal{Z},$$

i.e., the generator of the single particle evolving on $\mathcal{Z}$ under the static mean-field ξ.

Let $C_0^1([0,T]\times\mathcal{Z})$ denote the space of real-valued functions on $[0,T]\times\mathcal{Z}$ with compact support that are continuously differentiable in the first argument. Given a trajectory $\varphi \in D([0,T],\mathcal{M}_1(\mathcal{Z}))$ such that the mapping $[0,T]\ni t\mapsto\varphi_t\in\mathcal{M}_1(\mathcal{Z})$ is absolutely continuous (see Dawson and Gärtner [26, Section 4.1]), one can define $\dot\varphi_t\in\mathbb{R}^\infty$ for almost all $t\in[0,T]$ such that

$$\langle\varphi_t,f_t\rangle=\langle\varphi_0,f_0\rangle+\int_{[0,t]}\langle\dot\varphi_u,f_u\rangle du+\int_{[0,t]}\langle\varphi_u,\partial_u f_u\rangle du$$

holds for each $f\in C_0^1([0,T]\times\mathcal{Z})$ and $t\in[0,T]$.

For a set A let $\sim\!A$ denote the complement of A. For two numbers a and b, let $a\vee b$ denote maximum of a and b. For a metric space $\mathcal{S}$, let $\mathcal{B}(\mathcal{S})$ denote the Borel σ-field on $\mathcal{S}$. Finally, constants are denoted by C and their values may be different in each occurrence.

4.2.2 Process-level large deviations

We first recall the definition of the large deviation principle for a family of random variables indexed by one parameter.

Definition 4.1 (Large deviation principle)**.** Let $(\mathcal{S},d_0)$ be a metric space. We say that a family $\{X^N,N\geq 1\}$ of $\mathcal{S}$-valued random variables defined on a probability space $(\Omega,\mathcal{F},P)$ satisfies the large deviation principle with rate function $I:S\to[0,\infty]$ if

- (Compactness of level sets). For any $s\geq 0$, $\Phi(s):=\{x\in\mathcal{S}:I(x)\leq s\}$ is a compact subset of $\mathcal{S}$;
- (LDP lower bound). For any $\gamma>0$, $\delta>0$, and $x\in\mathcal{S}$, there exists $N_0\geq 1$ such that

$$P(d_0(X^N,x)<\delta)\geq\exp\{-N(I(x)+\gamma)\}$$

  for any $N\geq N_0$;
- (LDP upper bound). For any $\gamma>0$, $\delta>0$, and $s>0$, there exists $N_0\geq 1$ such that

$$P(d_0(X^N,\Phi(s))\geq\delta)\leq\exp\{-N(s-\gamma)\}$$

  for any $N\geq N_0$.

This definition is also used to study the large deviations of a family of probability measures. For each $N \geq 1$, let $P^N = P \circ (X^N)^{-1}$, the law of the random variable X_N on $(\mathcal{S}, d_0)$. We say that the family of probability measures $\{P^N, N \geq 1\}$ satisfies the LDP on $(\mathcal{S}, d_0)$ with rate function I if the sequence of $\mathcal{S}$-valued random variables $\{X^N, N \geq 1\}$ satisfies the LDP with rate function I.

The LDP lower bound in the above definition is equivalent to the following statement [37, Chapter 3, Section 3]

$$\liminf_{N\to\infty} \frac{1}{N} \log P(X^N \in G) \geq -\inf_{x\in G} I(x), \text{ for all } G \subset S \text{ open.}$$

Similarly, under the compactness of the level sets of the rate function I, the LDP upper bound above is equivalent to the following statement:

$$\limsup_{N\to\infty} \frac{1}{N} \log P(X^N \in F) \leq -\inf_{x\in F} I(x), \text{ for all } F \subset S \text{ closed.}$$

To study the LDP for the family of invariant measures, we require estimates on the probabilities of the process-level large deviations of μ_ν^N. In particular, we consider hitting times of μ_ν^N on certain subsets of the state space $\mathcal{M}_1(\mathcal{Z})$ and apply the process-level large deviation lower and upper bounds for μ_ν^N starting at these subsets. Therefore, in addition to the scaling parameter N, we must consider the process μ_ν^N indexed by the initial condition $\nu \in \mathcal{M}_1^N(\mathcal{Z})$. To study the process-level large deviations of such stochastic processes indexed by two parameters, we use the following definition of the uniform large deviation principle (see Freidlin and Wentzell [37, Chapter 3, Section 3]).

Definition 4.2 (Uniform large deviation principle)**.** Let $(\mathcal{S}, d_0)$ be a metric space. We say that a family $\{X_y^N, y \in \mathcal{S}, N \geq 1\}$ of $D([0,T],\mathcal{S})$-valued random variables defined on a probability space $(\Omega, \mathcal{F}, P)$ satisfies the uniform large derivation principle over the class $\mathcal{A}$ of subsets of $\mathcal{S}$ with the family of rate functions $\{I_y, y \in \mathcal{S}\}$, $I_y : D([0,T],\mathcal{S}) \to [0,+\infty]$, $y \in \mathcal{S}$, if

- (Compactness of level sets). For each $K \subset \mathcal{S}$ compact and $s \geq 0$, $\bigcup_{y\in K} \Phi_y(s)$ is a compact subset of $D([0,T],\mathcal{S})$, where $\Phi_y(s) := \{\varphi \in D([0,T],\mathcal{S}) : \varphi_0 = y, I_y(\varphi) \leq s\}$;
- (Uniform LDP lower bound). For any $\gamma > 0$, $\delta > 0$, $s > 0$, and $A \in \mathcal{A}$, there exists $N_0 \geq 1$ such that

  $$P(\rho(X_y^N, \varphi) < \delta) \geq \exp\{-N(I_y(\varphi) + \gamma)\},$$

  for all $y \in A$, $\varphi \in \Phi_y(s)$, and $N \geq N_0$;

- (Uniform LDP upper bound). For any $\gamma > 0$, $\delta > 0$, $s_0 > 0$, and $A \in \mathcal{A}$, there exists $N_0 \geq 1$ such that

$$P(\rho(X_y^N, \Phi_y(s)) \geq \delta) \leq \exp\{-N(s-\gamma)\},$$

for all $y \in A$, $s \leq s_0$, and $N \geq N_0$.

We now make some definitions. For each $\nu \in \mathcal{M}_1(\mathcal{Z})$ and $T > 0$, define the functional $S_{[0,T]}(\cdot|\nu) : D([0,T], \mathcal{M}_1(\mathcal{Z})) \to [0, \infty]$ by

$$S_{[0,T]}(\varphi|\nu) := \int_{[0,T]} \sup_{\alpha \in \mathbb{R}^\infty} \left\{ \langle \alpha, \dot{\varphi}_t - \Lambda^*_{\varphi_t}\varphi_t \rangle - \sum_{(z,z') \in \mathcal{E}} \tau(\alpha(z') - \alpha(z)) \lambda_{z,z'}(\varphi_t)\varphi_t(z) \right\} dt, \tag{4.5}$$

whenever $\varphi(0) = \nu$ and the mapping $[0,T] \ni t \mapsto \varphi_t \in \mathcal{M}_1(\mathcal{Z})$ is absolutely continuous; $S_{[0,T]}(\varphi|\nu) = \infty$ otherwise. Define the lower level sets of the functional $S_{[0,T]}(\cdot|\nu)$ by

$$\Phi_\nu^{[0,T]}(s) := \{\varphi \in D([0,T], \mathcal{M}_1(\mathcal{Z})) : \varphi_0 = \nu, S_{[0,T]}(\varphi|\nu) \leq s\}, \ s > 0, \ \nu \in \mathcal{M}_1(\mathcal{Z}).$$

The next lemma asserts that these level sets are compact in $D([0,T], \mathcal{M}_1(\mathcal{Z}))$ when the initial conditions belong to a compact subset of $\mathcal{M}_1(\mathcal{Z})$. The proof is deferred to Appendix 4.A.

Lemma 4.1. *For each $T > 0$, $s > 0$, and $K \subset \mathcal{M}_1(\mathcal{Z})$ compact,*

$$\{\varphi \in D([0,T], \mathcal{M}_1(\mathcal{Z})) : \varphi(0) \in K, S_{[0,T]}(\varphi|\varphi(0)) \leq s\}$$

is a compact subset of $D([0,T], \mathcal{M}_1(\mathcal{Z}))$.

The starting point of our study of the invariant measure asymptotics is the following uniform large deviation principle for the family $\{\mu_\nu^N, \nu \in \mathcal{M}_1^N(\mathcal{Z}), N \geq 1\}$ over the class of compact subsets of $\mathcal{M}_1(\mathcal{Z})$ with the family of rate functions $\{S_{[0,T]}(\cdot|\nu), \nu \in \mathcal{M}_1(\mathcal{Z})\}$. Its proof follows from the process-level LDP for μ_ν^N studied in Léonard [55] for a fixed initial condition and its extension to the case when initial conditions converge to a point in $\mathcal{M}_1(\mathcal{Z})$ in Borkar and Sundaresan [15]. The proof can be found in Appendix 4.A.

Theorem 4.2. *Fix $T > 0$ and assume (E1), (E2), and (E3). Then the family of $D([0,T], \mathcal{M}_1(\mathcal{Z}))$-valued random variables $\{(\mu_\nu^N(t), t \in [0,T]), \nu \in \mathcal{M}_1^N(\mathcal{Z}), N \geq 1\}$ satisfies the uniform large deviation principle over the class of compact subsets of $\mathcal{M}_1(\mathcal{Z})$ with the family of rate functions $\{S_{[0,T]}(\cdot|\nu), \nu \in \mathcal{M}_1(\mathcal{Z})\}$.*

The rate function $S_{[0,T]}(\cdot|\nu)$ admits a non-variational representation in terms of a minimal cost "control" that modulates the transition rates across various edges in $\mathcal{E}$ so that the desired trajectory is obtained.

Theorem 4.3 (Non-variational representation; Léonard [56]). *Let $\varphi \in D([0,T],\mathcal{M}_1(\mathcal{Z}))$ be such that $S_{[0,T]}(\varphi|\varphi(0)) < \infty$. Then there exists a measurable function $h_\varphi : [0,T]\times\mathcal{E} \to \mathbb{R}$ such that*

$$\langle \varphi_t, f_t\rangle = \langle \varphi_0, f_0\rangle + \int_{[0,t]} \langle \varphi_u, \partial_u f_u\rangle du + \int_{[0,t]} \sum_{(z,z')\in\mathcal{E}} (f_u(z') - f_u(z))(1 + h_\varphi(u,z,z'))\lambda_{z,z'}(\varphi_u)\varphi_u(z)du \tag{4.6}$$

holds for all $t \in [0,T]$ and all $f \in C_0^1([0,T]\times\mathcal{Z})$, and $S_{[0,T]}(\varphi|\varphi(0))$ admits the non-variational representation

$$S_{[0,T]}(\varphi|\varphi(0)) = \int_{[0,T]} \sum_{(z,z')\in\mathcal{E}} \tau^*(h_\varphi(t,z,z'))\lambda_{z,z'}(\varphi_t)\varphi_t(z)dt.$$

Remark 4.1. It can be shown that the rate function $S_{[0,T]}$ defined in (4.5) can also be expressed as

$$S_{[0,T]}(\varphi|\nu) = \sup_{f\in C_0^1([0,T]\times\mathcal{Z})} \Bigg\{ \langle \varphi_T, f_T\rangle - \langle \varphi_0, f_0\rangle - \int_{[0,T]} \langle \varphi_u, \partial_u f_u\rangle du - \int_{[0,T]} \langle \varphi_u, L_{\varphi_u} f_u\rangle du - \int_{[0,T]} \sum_{(z,z')\in\mathcal{E}} \tau(f_u(z') - f_u(z))\lambda_{z,z'}(\varphi_u)\varphi_u(z)du \Bigg\}, \tag{4.7}$$

$\varphi \in D([0,T],\mathcal{M}_1(\mathcal{Z}))$, see Léonard [56]. This form of the rate function will indeed be used in the proof of the counterexamples in Section 4.8.

4.3 Invariant measure: Existence, uniqueness, and exponential tightness

In this section we prove Proposition 4.1, the existence and uniqueness of the invariant measure $\wp^N$ for $\mathscr{L}^N$ for each $N \geq 1$, and the exponential tightness of the family of invariant measures $\{\wp^N, N \geq 1\}$. The proof relies on the standard Krylov-Bogolyubov argument and a coupling between the interacting particle system under consideration and a non-interacting system with maximal forward transition rates minimal backward transition rates.

We first introduce some notations for the non-interacting particle system. Let $\bar{L}$ denote the generator acting on functions f on $\mathcal{Z}$ by $f \mapsto \sum_{z':(z,z')\in\mathcal{E}}(f(z') - f(z))\lambda_{z,z'}$, $z \in \mathcal{Z}$, where $\lambda_{z,z+1} = \overline{\lambda}/(z+1)$ and $\lambda_{z,0} = \underline{\lambda}$. For each $z \in \mathcal{Z}$, let $\bar{P}_z$ denote the solution to the $D([0,T],\mathcal{Z})$-valued martingale problem for $\bar{L}$ with initial condition z. Integration with respect to $\bar{P}_z$ is denoted by $\bar{E}_z$. Let $\pi \in \mathcal{M}_1(\mathcal{Z})$ denote the unique invariant probability measures for $\bar{L}$. Integration with respect to π is denoted by $\bar{E}_\pi$. Finally, for each $N \geq 1$, let $\bar{\mathbb{P}}^N_\nu$ denote the solution to the $D([0,T],\mathcal{M}_1^N(\mathcal{Z}))$-valued martingale problem for $\mathscr{L}^N$ with initial condition ν, $\lambda_{z,z+1}(\zeta)$ replaced by $\overline{\lambda}/(z+1)$ and $\lambda_{z,0}(\zeta)$ replaced by $\underline{\lambda}$ in (4.1), respectively, for each $\zeta \in \mathcal{M}_1(\mathcal{Z})$. Integration with respect to $\bar{\mathbb{P}}^N_\nu$ is denoted by $\bar{\mathbb{E}}^N_\nu$. We are now ready to prove Proposition 4.1.

Proof of Proposition 4.1. Fix $N \geq 1$. We first show the existence and uniqueness of the invariant probability measure for $\mathscr{L}^N$. Consider the family of probability measures $\{\eta^N_T, T \geq 1\}$ on $\mathcal{M}_1(\mathcal{Z})$ defined by

$$\eta^N_T(A) := \frac{1}{T}\int_0^T \mathbb{P}^N_{\delta_0}(\mu^N(t) \in A)dt,\ A \in \mathcal{B}(\mathcal{M}_1(\mathcal{Z})),\ T \geq 1.$$

Let $X^N_n(t)$ denote the state of the nth particle at time t. Note that, for any $t > 1$, $M > 1$, and $\beta > 0$,

$$\begin{aligned}
\mathbb{P}^N_{\delta_0}(\mu^N(t) \notin \mathscr{K}_M) &\leq \bar{\mathbb{P}}^N_{\delta_0}(\mu^N(t) \notin \mathscr{K}_M)\\
&= \bar{\mathbb{P}}^N_{\delta_0}\left(\sum_{n=1}^N \vartheta(X^N_n(t)) > NM\right)\\
&\leq \exp\{-NM\beta\}\bar{\mathbb{E}}^N_{\delta_0}\left(\exp\left\{\beta\sum_{n=1}^N \vartheta(X^N_n(t))\right\}\right)\\
&= \exp\{-NM\beta\}(\bar{E}_0(\exp\{\beta\vartheta(X^N_1(t))\}))^N,
\end{aligned} \tag{4.8}$$

where the first inequality follows from a straightforward coupling between the evolution of each particle under $\mathbb{P}^N_{\delta_0}$ and $\bar{\mathbb{P}}^N_{\delta_0}$, and the second inequality is a consequence of Chebyshev's inequality. Note that, again by a coupling argument, $\bar{E}_0(\exp\{\beta\vartheta(X^N_1(t))\}) \leq \bar{E}_\pi(\exp\{\beta\vartheta(X^N_1(t))\})$. The latter is finite for sufficiently small $\beta > 0$, thanks to the $\exp\{-\vartheta(z)\}$ decay of the probability measure π on $\mathcal{Z}$. Thus we can choose $\bar{\beta} > 0$ small enough (independent of M) so that

$\log \bar{E}_\pi(\exp\{\bar{\beta}\vartheta(X_1^N(t))\}) < 1$. Hence (4.8) implies that

$$\mathbb{P}_{\delta_0}^N(\mu^N(t) \notin \mathscr{K}_M) \leq \exp\{-N(M\bar{\beta} - 1)\}.$$

Therefore, for any $M > 0$ and $T \geq 1$, we get

$$\eta_T^N(\sim\mathscr{K}_M) \leq \exp\{-N(M\bar{\beta} - 1)\}. \tag{4.9}$$

Since $\mathscr{K}_M$ is a compact subset of $\mathcal{M}_1(\mathcal{Z})$, this show that the family $\{\eta_T^N, T \geq 1\}$ is tight. Hence it follows that there exists an invariant probability measure $\wp^N$ for $\mathscr{L}^N$ (see, for example, Ethier and Kurtz [34, Theorem 9.3, page 240]). By Assumption (E1), μ^N is an irreducible Markov process; hence $\wp^N$ is the unique invariant probability measure for $\mathscr{L}^N$.

We now show the exponential tightness of the family $\{\wp^N, N \geq 1\}$. Let $M > 0$ be given, and choose $M' = (M+1)/\bar{\beta}$. For each $N \geq 1$, since $\wp^N$ is a weak limit of the family $\{\eta_T^N, T \geq 1\}$ as $T \to \infty$, from (4.9) with M replaced by M', it follows that

$$\wp^N(\sim\mathscr{K}_{M'}) \leq \liminf_{T\to\infty} \eta_T^N(\sim\mathscr{K}_{M'}) \leq \exp\{-NM\}. \tag{4.10}$$

for each $N \geq 1$. Hence,

$$\limsup_{N\to\infty} \frac{1}{N} \log \wp^N(\sim\mathscr{K}_{M'}) \leq -M,$$

which establishes that the family $\{\wp^N, N \geq 1\}$ is exponential tight. This completes the proof of the proposition. □

4.4 The LDP lower bound

In this section we prove the LDP lower bound for the family $\{\wp^N, N \geq 1\}$. To lower bound the probability of a small neighbourhood of a point ξ under $\wp^N$, we first produce a trajectory that starts at $\mathscr{K}_M$ for a suitable $M > 0$, connects to ξ^* with a small cost, and then reaches ξ from ξ^* with cost arbitrarily close to $V(\xi)$. The probability of a small neighbourhood of ξ under $\wp^N$ is then lower bounded by the probability that the process μ^N remains in a small neighbourhood of the trajectory constructed above. The latter is then lower bounded using the uniform LDP lower bound for μ^N, where the uniformity is over the initial condition lying in a given compact subset of $\mathcal{M}_1(\mathcal{Z})$.

We begin with a lemma that allows us to connect points in $K(\Delta)$ to ξ^* for small enough Δ

with small cost. We omit its proof here, since it follows from a certain continuity property of V which will be shown in Lemma 4.6.

Lemma 4.2. *Given $\gamma > 0$ there exists $\Delta > 0$ such that for any $\zeta \in K(\Delta)$ there exists a $T > 0$ and a trajectory φ on $[0, T]$ such that $\varphi(0) = \zeta$, $\varphi(T) = \xi^*$, and $S_{[0,T]}(\varphi|\zeta) \leq \gamma$.*

We now prove the LDP lower bound for the family $\{\wp^N, N \geq 1\}$.

Lemma 4.3. *For any $\gamma > 0$, $\delta > 0$, and $\xi \in \mathcal{M}_1(\mathcal{Z})$, there exists $N_0 \geq 1$ such that*

$$\wp^N\{\zeta \in \mathcal{M}_1(\mathcal{Z}) : d(\zeta, \xi) < \delta\} \geq \exp\{-N(V(\xi) + \gamma)\} \tag{4.11}$$

for all $N \geq N_0$.

Proof. Fix $\gamma > 0$, $\delta > 0$, and $\xi \in \mathcal{M}_1(\mathcal{Z})$. We may assume that $V(\xi) < \infty$; if $V(\xi) = \infty$ then (4.11) trivially holds for all $N \geq 1$. Choose some $M > 0$ and $N_1 \geq 1$ such that $\wp^N(\mathscr{K}_M) \geq 1/2$ for all $N \geq N_1$; this is possible from the exponential tightness of the family $\{\wp^N, N \geq 1\}$, see Proposition 4.1. Using Lemma 4.2, choose $\varepsilon > 0$ and $T_0 > 0$ such that for any $\zeta_1 \in K(\varepsilon)$ there exists a trajectory φ_1 on $[0, T_0]$ such that $\varphi_1(0) = \zeta_1, \varphi_1(T_0) = \xi^*$, and $S_{[0,T_0]}(\varphi_1|\zeta_1) \leq \gamma/4$. Since ξ^* is the globally asymptotically stable equilibrium for (4.2), for the above $\varepsilon > 0$, there exists a $T_1 > 0$ such that for any $\zeta \in \mathscr{K}_M$ we have $\mu_\zeta(T_1) \in K(\varepsilon)$, where μ_ζ denotes the solution to the McKean-Vlasov equation (4.2) with initial condition ζ (see assumption (F2)). Also, by the definition of $V(\xi)$, there exists a $T_2 > 0$ and a trajectory φ_2 such that $\varphi_2(0) = \xi^*, \varphi_2(T_2) = \xi$, and $S_{[0,T_2]}(\varphi_2|\xi^*) \leq V(\xi) + \gamma/4$. Let $T = T_1 + T_0 + T_2$. Given $\zeta \in \mathscr{K}_M$, we construct a trajectory φ_ζ on $[0, T]$ by using the above three trajectories as follows. Let $\varphi_\zeta(0) = \zeta$; $\varphi_\zeta(t) = \mu_\zeta(t)$ for $t \in [0, T_1]$; $\varphi_\zeta(t) = \varphi_1(t - T_1)$ for $t \in (T_1, T_1 + T_0]$; and $\varphi_\zeta(t) = \varphi_2(t - (T_1 + T_0))$ for $t \in (T_1 + T_0, T]$. Note that $S_{[0,T]}(\varphi_\zeta|\zeta) \leq V(\xi) + \gamma/2$. From the uniform continuity of φ_ζ on $[0, T]$, we can choose $\delta' > 0$ such that $\rho(\varphi, \varphi_\zeta) < \delta'$ implies $d(\varphi(T), \varphi_\zeta(T)) < \delta$ for any $\varphi \in D([0, T], \mathcal{M}_1(\mathcal{Z}))$. Then for each $N \geq N_1$, we have

$$\begin{aligned}
\wp^N\{\zeta \in \mathcal{M}_1(\mathcal{Z}) : d(\zeta, \xi) < \delta\} &= \int_{\mathcal{M}_1^N(\mathcal{Z})} \mathbb{P}_\zeta^N(d(\mu^N(T), \xi) < \delta)\wp^N(d\zeta) \\
&\geq \int_{\mathscr{K}_M \cap \mathcal{M}_1^N(\mathcal{Z})} \mathbb{P}_\zeta^N(d(\mu^N(T), \xi) < \delta)\wp^N(d\zeta) \\
&\geq \int_{\mathscr{K}_M \cap \mathcal{M}_1^N(\mathcal{Z})} \mathbb{P}_\zeta^N(\rho(\mu^N, \varphi_\zeta) < \delta')\wp^N(d\zeta) \\
&\geq \frac{1}{2} \inf_{\zeta \in \mathscr{K}_M \cap \mathcal{M}_1^N(\mathcal{Z})} \mathbb{P}_\zeta^N(\rho(\mu^N, \varphi_\zeta) < \delta'); \qquad (4.12)
\end{aligned}$$

here the first equality follows since $\wp^N$ is invariant to time shifts. By the uniform LDP lower bound in Theorem 4.2, there exists $N_2 \geq N_1$ such that

$$\mathbb{P}_\zeta^N(\rho(\mu^N, \varphi) < \delta') \geq \exp\{-N(S_{[0,T]}(\varphi|\zeta) + \gamma/4)\}$$

for all $\zeta \in \mathscr{K}_M \cap \mathcal{M}_1^N(\mathcal{Z})$, $\varphi \in \Phi_\zeta^{[0,T]}(V(\xi) + \gamma/2)$, and $N \geq N_2$. Noting that $S_{[0,T]}(\varphi_\zeta|\zeta) \leq V(\xi) + \gamma/2$ for any $\zeta \in \mathscr{K}_M \cap \mathcal{M}_1^N(\mathcal{Z})$, and using the above uniform LDP lower bound, (4.12) becomes

$$\wp^N\{\zeta \in \mathcal{M}_1(\mathcal{Z}) : d(\zeta, \xi) < \delta\} \geq \frac{1}{2}\exp\{-N(V(\xi) + 3\gamma/4)\}$$

for all $N \geq N_2$. Finally, choose $N_0 \geq N_2$ so that $1/2 \geq \exp\{-N\gamma/4\}$. Then the above becomes

$$\wp^N\{\zeta \in \mathcal{M}_1(\mathcal{Z}) : d(\zeta, \xi) < \delta\} \geq \exp\{-N(V(\xi) + \gamma)\}$$

for all $N \geq N_0$. This completes the proof of LDP lower bound for the family $\{\wp^N, N \geq 1\}$. $\square$

4.5 Properties of the quasipotential

In this section we prove three key properties of the quasipotential V. These three properties are (i) a characterisation of the set of points for which V is finite, (iii) a certain continuity property for V, and (iii) the compactness of the lower level sets of V. These properties play an important role in the proof of the LDP upper bound in Section 4.6.

4.5.1 A characterisation of finiteness of the quasipotential

Recall the function ϑ and the compact sets $\mathscr{K}_M$, $M > 0$. We start with a lemma that enables us to connect δ_0, the point mass at state 0, to a point $\xi \in \mathscr{K}_M$ for some $M > 0$. This connection is made using a piecewise constant velocity trajectory wherein for each $z \geq 1$, we move the mass $\xi(z)$ from state 0 to state z in z steps; in the kth step, we move the mass $\xi(z)$ from state $k-1$ to state k with unit velocity. The lemma asserts that the cost of this piecewise constant velocity trajectory is bounded above by a constant that depends only on M.

Lemma 4.4. *Given $M > 0$ there exists a constant C_M depending on M such that for any $\xi \in \mathscr{K}_M$ there exists a $T > 0$ and a trajectory φ on $[0, T]$ such that $\varphi(0) = \delta_0$, $\varphi(T) = \xi$, and $S_{[0,T]}(\varphi|\delta_0) \leq C_M$.*

Proof. Fix $M > 0$ and $\xi \in \mathscr{K}_M$. Fix $J \in \mathcal{Z} \setminus \{0\}$ and define $\mathcal{Z}_J = \{1, 2, \dots, J\}$, $t_z = z\xi(z)$ for $z \in \mathcal{Z}_J$, and $T_z = \sum_{z' \in \mathcal{Z}_J, z' \geq z} t_{z'}$. Note that $T_J \leq T_{J-1} \leq \cdots \leq T_1$. We shall first construct a trajectory φ^J such that $\varphi^J(0) = \delta_0$, $\varphi^J(T_1)(z) = \xi(z)$ for each $z \in \mathcal{Z}_J$, and $S_{[0,T_1]}(\varphi^J|\delta_0)$ bounded above by a constant independent of J.

Let $T_{J+1} = 0$. For each $z \in \mathcal{Z}_J$, starting with $z = J$, we move the mass $\xi(z)$ from the state 0 to state z using a piecewise unit velocity trajectory over the time duration $(T_{z+1}, T_{z+1} + t_z]$. We define this trajectory φ^J on $[0, T_1]$ as follows. Let $\varphi_0^J = \delta_0$. For each $z \in \mathcal{Z}_J$ and $1 \leq k \leq z$, when $t \in (T_{z+1} + (k-1)\xi(z), T_{z+1} + k\xi(z)]$, let

$$\dot{\varphi}_t^J(l) = \begin{cases} 1 & \text{if } l = k \\ -1 & \text{if } l = k-1 \\ 0 & \text{otherwise,} \end{cases}$$

$l \in \mathcal{Z}$, and define $\varphi_t^J(l) = \delta_0(l) + \int_{[0,t]} \dot{\varphi}_u^J(l)du$, $l \in \mathcal{Z}$, $t \in [0, T]$.

We now calculate the cost of this trajectory. For a fixed $z \in \mathcal{Z}$ and $1 \leq k \leq z$, for each $t \in (T_{z+1} + (k-1)\xi(z), T_{z+1} + k\xi(z)]$ and $\alpha \in \mathbb{R}^\infty$, note that

$$\begin{aligned}
&\langle \alpha, \dot{\varphi}_t^J - \Lambda_{\varphi_t^J}^* \varphi_t^J \rangle - \sum_{(z,z') \in \mathcal{E}} \tau(\alpha(z') - \alpha(z))\lambda_{z,z'}(\varphi_t^J)\varphi_t^J(z) \\
&\quad = (\alpha(k) - \alpha(k-1)) - \sum_{(z,z') \in \mathcal{E}} (\exp\{\alpha(z') - \alpha(z)\} - 1)\lambda_{z,z'}(\varphi_t^J)\varphi_t^J(z).
\end{aligned}$$

Hence,

$$\begin{aligned}
&\sup_{\alpha \in \mathbb{R}^\infty} \left\{ \langle \alpha, \dot{\varphi}_t^J - \Lambda_{\varphi_t^J}^* \varphi_t^J \rangle - \sum_{(z,z') \in \mathcal{E}} \tau(\alpha(z') - \alpha(z))\lambda_{z,z'}(\varphi_t^J)\varphi_t^J(z) \right\} \\
&\quad \leq \sup_{x \in \mathbb{R}} (x - (\exp\{x\} - 1)\lambda_{k-1,k}(\varphi_t^J)\varphi_t^J(k-1)) \\
&\qquad + \sup_{\alpha \in \mathbb{R}^\infty} \left(- \sum_{(z,z') \in \mathcal{E}; (z,z') \neq (k-1,k)} (\exp\{\alpha(z') - \alpha(z)\} - 1)\lambda_{z,z'}(\varphi_t^J)\varphi_t^J(z) \right) \\
&\quad \leq \log \left(\frac{1}{\varphi_t^J(k-1)\lambda_{k-1,k}(\varphi_t^J)} \right) + 2\bar{\lambda} \\
&\quad \leq \log \left(\frac{1}{\varphi_t^J(k-1)} \right) + \log k + \log \left(\frac{1}{\underline{\lambda}} \right) + 2\bar{\lambda}, \qquad (4.13)
\end{aligned}$$

where the last two inequalities follow from assumption (E2). Consider the first term above. For $k > 1$, integration of this quantity over the time duration $t \in (T_{z+1} + (k-1)\xi(z), T_{z+1} + k\xi(z)]$

gives

$$\int_{(T_{z+1}+(k-1)\xi(z),T_{z+1}+k\xi(z)]} \log\left(\frac{1}{\varphi_t^J(k-1)}\right) dt = -\int_{\xi(z)}^{0} \log\left(\frac{1}{u}\right) du$$
$$= (u\log u - u)\Big|_{\xi(z)}^{0}$$
$$= \xi(z)\log\left(\frac{1}{\xi(z)}\right) + \xi(z),$$

where the first equality follows from the variable change $u = \varphi_t^J(k-1)$ and the facts (i) $\dot{\varphi}_t^J(k-1) = -1$, (ii) $\varphi_t^J(k-1) = \xi(z)$ when $t = T_{z+1} + (k-1)\xi(z)$, (iii) $\varphi_t^J(k-1) = 0$ when $t = T_{z+1} + k\xi(z)$, and (iv) $du = -dt$. For $k = 1$, using the bound $\varphi_t^J(0) \geq \varphi_t^J(0) - (1 - \sum_{z' \geq z} \xi(z'))$, we get

$$\int_{(T_{z+1},T_{z+1}+\xi(z)]} \log\left(\frac{1}{\varphi_t^J(0)}\right) dt$$
$$\leq \int_{t\in(T_{z+1},T_{z+1}+\xi(z)]} \log\left(\frac{1}{\varphi_t^J(0) - (1 - \sum_{z'\geq z}\xi(z'))}\right) dt$$
$$= -\int_{\xi(z)}^{0} \log\left(\frac{1}{u}\right) du,$$

where the last equality follows from the variable change $u = \varphi_t^J(0) - (1 - \sum_{z' \geq z} \xi(z'))$, and the facts (i) $\varphi_t^J(0) = -1$, (ii) $\varphi_t^J(0) = 1 - \sum_{z'>z} \xi(z')$ when $t = T_{z+1}$ so that $\varphi_t^J(0) - (1 - \sum_{z'\geq z} \xi(z')) = \xi(z)$ when $t = T_{z+1}$, (iii) $\varphi_t^J(0) = 1 - \sum_{z'\geq z} \xi(z')$ when $t = T_{z+1} + \xi(z)$ so that $\varphi_t^J(0) - (1 - \sum_{z'\geq z} \xi(z')) = 0$ when $t = T_{z+1} + \xi(z)$, and (iv) $du = -dt$. Thus, proceeding as before for the case $k > 1$, we arrive at

$$\int_{(T_{z+1},T_{z+1}+\xi(z)]} \log\left(\frac{1}{\varphi_t^J(0)}\right) dt \leq \xi(z)\log\left(\frac{1}{\xi(z)}\right) + \xi(z).$$

Hence, integrating (4.13) over $t \in (T_{z+1}+(k-1)\xi(z), T_{z+1}+k\xi(z)]$ and summing over $1 \leq k \leq z$, we get, for each $z \in \mathcal{Z}_J$,

$$\int_{(T_{z+1},T_{z+1}+t_z]} \sup_{\alpha\in\mathbb{R}^\infty} \left\{ \langle \alpha, \dot{\varphi}_t^J - \Lambda^*_{\varphi_t^J}\varphi_t^J \rangle - \sum_{(z,z')\in\mathcal{E}} \tau(\alpha(z') - \alpha(z))\lambda_{z,z'}(\varphi_t^J)\varphi_t^J(z) \right\} dt$$
$$\leq z\xi(z)\log\left(\frac{1}{\xi(z)}\right) + \tilde{C}_z$$

where $\tilde{C}_z = (z\log z + z)\xi(z) + z\xi(z)\left(\log\left(\frac{1}{\underline{\lambda}}\right) + 2\overline{\lambda}\right)$. Let $\tilde{C}^J = \sum_{z\in\mathcal{Z}_J}\tilde{C}_z$. Thus, summing the above display over $z\in\mathcal{Z}_J$, we arrive at

$$S_{[0,T_1]}(\varphi^J|\delta_0) \le \sum_{z\in\mathcal{Z}_J} z\xi(z)\log\left(\frac{1}{\xi(z)}\right) + \tilde{C}^J.$$

Note that

$$\begin{aligned}\sum_{z\in\mathcal{Z}_J} z\xi(z)\log\left(\frac{1}{\xi(z)}\right) &= \sum_{\substack{z\in\mathcal{Z}_J:\\ \xi(z)\le 1/z^3}} z\xi(z)\log\left(\frac{1}{\xi(z)}\right) + \sum_{\substack{z\in\mathcal{Z}_J:\\ \xi(z)> 1/z^3}} z\xi(z)\log\left(\frac{1}{\xi(z)}\right)\\ &\le \frac{1}{e} + \sum_{\substack{z\in\mathcal{Z}_J\setminus\{1\}:\\ \xi(z)\le 1/z^3}} \frac{3\log z}{z^2} + 3\sum_{\substack{z\in\mathcal{Z}_J:\\ \xi(z)> 1/z^3}} z\log z\xi(z)\\ &\le \frac{1}{e} + 3\sum_{z\in\mathcal{Z}_J}\left\{\frac{\log z}{z^2} + z\log z\xi(z)\right\},\end{aligned} \tag{4.14}$$

where the first inequality comes from the fact that the mapping $x\mapsto x\log(1/x)$ is monotonically increasing for $x\in[0,1/e]$. Hence,

$$S_{[0,T_1]}(\varphi^J|\delta_0) \le \frac{1}{e} + 3\sum_{z\in\mathcal{Z}_J}\left\{\frac{\log z}{z^2} + z\log z\xi(z)\right\} + \tilde{C}^J,\ J\ge 1.$$

Define $T = \sum_{z\in\mathcal{Z}} z\xi(z)$. We now extend the trajectory φ^J to $(T_1, T]$ by defining $\varphi_t^J = \varphi_{T_1}^J$ for $t\in(T_1,T]$. Noting that $\dot{\varphi}_t^J(z) = 0$ for all $z\in\mathcal{Z}$ on $t\in(T_1,T]$, this extension suffers an additional cost of at most $2\overline{\lambda}T$. Hence, we get

$$S_{[0,T]}(\varphi^J|\delta_0) \le \frac{1}{e} + 3\sum_{z\in\mathcal{Z}_J}\left\{\frac{\log z}{z^2} + z\log z\xi(z)\right\} + \tilde{C}^J + 2\overline{\lambda}T.$$

Noting that (i) the right hand side above is upper bounded by $\langle\xi,\vartheta\rangle C(\overline{\lambda},\underline{\lambda})$, where $C(\overline{\lambda},\underline{\lambda})$ is a constant depending on $\overline{\lambda}$ and $\underline{\lambda}$, and (ii) $\langle\xi,\vartheta\rangle\le M$, the above display yields

$$S_{[0,T]}(\varphi^J|\delta_0) \le C(M,\overline{\lambda},\underline{\lambda}),$$

where $C(M,\overline{\lambda},\underline{\lambda})$ is a constant depending on $M,\overline{\lambda}$, and $\underline{\lambda}$. Using the compactness of the level sets of $S_{[0,T]}$ (see Lemma 4.1), it follows that the sequence of trajectories $\{\varphi^J, J\ge 1\}$ has a convergent subsequence. Re-indexing the original sequence, let $\varphi^J\to\varphi$ in $D([0,T],\mathcal{M}_1(\mathcal{Z}))$ as $J\to\infty$. By construction, for each $J\in\mathcal{Z}\setminus\{0\}$, $\varphi_T^J(z) = \xi(z)$ for all $z\in\mathcal{Z}_J$; hence $\varphi_T(z) = \xi(z)$

for all $z \in \mathcal{Z}$. By the lower semicontinuity of $S_{[0,T]}$, it follows that

$$S_{[0,T]}(\varphi|\delta_0) \le \liminf_{J\to\infty} S_{[0,T]}(\varphi^J|\delta_0) \le C(M, \overline{\lambda}, \underline{\lambda}).$$

This completes the proof of the lemma. □

We are now ready to characterise the set of points ξ in $\mathcal{M}_1(\mathcal{Z})$ whose $V(\xi)$ is finite.

Lemma 4.5. *$V(\xi) < \infty$ if and only if $\xi \in \mathscr{K}$. Furthermore, for any $M > 0$, there exists a constant $C_M > 0$ such that $\xi \in \mathscr{K}_M$ implies $V(\xi) \le C_M$.*

Proof. Let $\xi \in \mathcal{M}_1(\mathcal{Z})$ be such that $V(\xi) < \infty$. Then there exists a $T > 0$ and a trajectory φ on $[0, T]$ such that $\varphi(0) = \xi^*, \varphi(T) = \xi$, and $S_{[0,T]}(\varphi|\xi^*) \le V(\xi) + 1$. By Theorem 4.3, there exists a measurable function h_φ on $[0, T] \times \mathcal{E}$ such that

$$\langle \varphi_t, f\rangle = \langle \varphi_0, f\rangle + \int_{[0,t]} \sum_{(z,z')\in\mathcal{E}} (f(z') - f(z))(1 + h_\varphi(u, z, z'))\lambda_{z,z'}(\varphi_u)\varphi_u(z)du \tag{4.15}$$

holds for all $t \in [0, T]$ and $f \in C_0(\mathcal{Z})$, and $S_{[0,T]}(\varphi|\varphi(0))$ is given by

$$S_{[0,T]}(\varphi|\varphi(0)) = \int_{[0,T]} \sum_{(z,z')\in\mathcal{E}} \tau^*(h_\varphi(t, z, z'))\lambda_{z,z'}(\varphi_t)\varphi_t(z)dt.$$

For any $x \ge 0$ and $y \in \mathbb{R}$, using the convex duality relation $(x - 1)y \le \tau^*(x - 1) + \tau(y)$, we get the inequality $xy \le \tau^*(x - 1) + (\exp\{y\} - 1)$. Hence, from the above non-variational representation for $S_{[0,T]}(\varphi|\varphi(0))$, (4.15) implies

$$\begin{aligned}\langle \varphi_t, f\rangle &\le \langle \xi^*, f\rangle + \int_{[0,t]} \sum_{(z,z')\in\mathcal{E}} \tau^*(h_\varphi(u, z, z'))\lambda_{z,z'}(\varphi_u)\varphi_u(z)du \\ &\qquad + \int_{[0,t]} \sum_{(z,z')\in\mathcal{E}} (\exp\{f(z') - f(z)\} - 1)\lambda_{z,z'}(\varphi_u)\varphi_u(z)du \\ &\le \langle \xi^*, f\rangle + V(\xi) + 1 \\ &\qquad + \int_{[0,t]} \sum_{(z,z')\in\mathcal{E}} (\exp\{f(z') - f(z)\} - 1)\lambda_{z,z'}(\varphi_u)\varphi_u(z)du. \end{aligned} \tag{4.16}$$

Recall the function ϑ on $\mathcal{Z}$. For $n \geq 1$, define

$$\vartheta_n(z) = \begin{cases} \vartheta(z), & \text{if } z \leq n, \\ 0, & \text{otherwise.} \end{cases}$$

By convexity, note that $\vartheta_n(z+1) - \vartheta_n(z) \leq 1 + \log(z+1)$ and $\vartheta_n(0) - \vartheta_n(z) \leq 0$, for each $z \in \mathcal{Z}$. Therefore, using the upper bound for the transition rates from assumption (E2), observe that

$$\int_{[0,t]} \sum_{(z,z') \in \mathcal{E}} (\exp\{\vartheta_n(z') - \vartheta_n(z)\} - 1)\lambda_{z,z'}(\varphi_u)\varphi_u(z)du \leq \bar{\lambda}(e-1)t,$$

for each $t \in [0,T]$ and $n \geq 1$. It follows from (4.16) with f replaced by ϑ_n that

$$\langle \varphi_t, \vartheta_n \rangle \leq \langle \xi^*, \vartheta_n \rangle + V(\xi) + 1 + \bar{\lambda}(e-1)T$$

for each $t \in [0,T]$ and $n \geq 1$. Letting $n \to \infty$ and using monotone convergence, we conclude that

$$\sup_{t \in [0,T]} \langle \varphi_t, \vartheta \rangle = \sup_{t \in [0,T]} \lim_{n \to \infty} \langle \varphi_t, \vartheta_n \rangle \leq \langle \xi^*, \vartheta \rangle + V(\xi) + 1 + \bar{\lambda}(e-1)T. \tag{4.17}$$

In particular, $\langle \xi, \vartheta \rangle \leq \langle \xi^*, \vartheta \rangle + V(\xi) + 1 + \bar{\lambda}(e-1)T$. It follows that $\xi \in \mathscr{K}$.

Conversely, let $\xi \in \mathscr{K}$. Let $M > 0$ be such that $\xi \in \mathscr{K}_M$. By Lemma 4.4, there exists a $T > 0$ and a trajectory $\varphi^{(2)}$ on $[0,T]$ such that $\varphi^{(2)}(0) = \delta_0$, $\varphi^{(2)}(T) = \xi$, and $S_{[0,T]}(\varphi^{(2)}|\delta_0) \leq C_M$ for some constant $C_M > 0$ depending on M. Let $t_0 = 0$, $t_z = \sum_{z'=1}^{z} \xi(z')$, $z \in \mathcal{Z} \setminus \{0\}$, and $T_1 = \sum_{z' \neq 0} \xi(z')$. We construct another trajectory $\varphi^{(1)}$ on $[0,T_1]$ such that $\varphi^{(1)}(0) = \xi^*$, $\varphi^{(1)}(T_1) = \delta_0$, and $S_{[0,T_1]}(\varphi^{(1)}|\xi^*) < \infty$ as follows. When $t \in (t_{z-1}, t_z]$ for some $z \in \mathcal{Z} \setminus \{0\}$, let

$$\dot{\varphi}_t^{(1)}(l) = \begin{cases} -1, & \text{if } l = z, \\ 1, & \text{if } l = 0, \\ 0, & \text{otherwise,} \end{cases}$$

$l \in \mathcal{Z}$, and define $\varphi_t^{(1)}(t) = \varphi_0^{(1)}(l) + \int_{[0,t]} \dot{\varphi}_u^{(1)}(l)du$, $l \in \mathcal{Z}$, $t \in [0,T_1]$. Note that, for each $\alpha \in \mathbb{R}^\infty$, when $t \in (t_{z-1}, t_z]$ for some $z \in \mathcal{Z} \setminus \{0\}$, we have

$$\begin{aligned} &\left\{ \langle \alpha, \dot{\varphi}_t^{(1)} - \Lambda^*_{\varphi_t^{(1)}} \varphi_t^{(1)} \rangle - \sum_{(z,z') \in \mathcal{E}} \tau(\alpha(z') - \alpha(z))\lambda_{z,z'}(\varphi_t^{(1)})\varphi_t^{(1)}(z) \right\} \\ &\quad = (\alpha(0) - \alpha(z)) - (\exp\{\alpha(0) - \alpha(z)\} - 1)\lambda_{z,0}(\varphi_t^{(1)})\varphi_t^{(1)}(z) \end{aligned}$$

$$- \sum_{(z_0,z')\in\mathcal{E}:(z_0,z')\neq(z,0)} (\exp\{\alpha(z')-\alpha(z_0)\}-1)\lambda_{z_0,z'}(\varphi_t^{(1)})\varphi_t^{(1)}(z_0)\Bigg\},$$

so that optimising the left hand side of the above display over $\alpha \in \mathbb{R}^\infty$ yields

$$\begin{aligned}
\sup_{\alpha\in\mathbb{R}^\infty}&\left\{\langle\alpha,\dot{\varphi}_t^{(1)} - \Lambda^*_{\varphi_t^{(1)}}\varphi_t^{(1)}\rangle - \sum_{(z,z')\in\mathcal{E}} \tau(\alpha(z')-\alpha(z))\lambda_{z,z'}(\varphi_t^{(1)})\varphi_t^{(1)}(z)\right\}\\
&\leq \log\left(\frac{1}{\varphi_t^{(1)}(z)\lambda_{z,0}(\varphi_t^{(1)})}\right) + 2\bar{\lambda}\\
&\leq \log\left(\frac{1}{\varphi_t^{(1)}(z)}\right) + \log\left(\frac{1}{\underline{\lambda}}\right) + 2\bar{\lambda},
\end{aligned}$$

where the last inequality follows form the lower bound on the backward transition rates in assumption (E2). Integrating the above over $(t_{z-1}, t_z]$ and summing over $z \in \mathcal{Z}\setminus\{0\}$, we arrive at

$$S_{[0,T_1]}(\varphi^{(1)}|\xi) \leq \sum_{z\in\mathcal{Z}\setminus\{0\}} \left\{\xi^*(z)\log\frac{1}{\xi^*(z)} + \xi^*(z)\left(\log\left(\frac{1}{\underline{\lambda}}\right) + 2\bar{\lambda}\right)\right\}.$$

Since $\xi^* \in \mathscr{K}$, proceeding via the steps in (4.14), we conclude that the right hand side of the above display is finite. We combine $\varphi^{(1)}$ and $\varphi^{(2)}$ and define a new trajectory $\tilde{\varphi}$ on $[0, T_1 + T]$ as follows: $\tilde{\varphi}(t) = \varphi^{(1)}(t)$ on $t \in [0, T_1]$; $\tilde{\varphi}(t) = \varphi^{(2)}(t - T_1)$ on $t \in (T_1, T_1 + T]$. Note that $\tilde{\varphi}(0) = \xi^*$, $\tilde{\varphi}(T_1 + T) = \xi$, and $S_{[0,T_1+T]}(\tilde{\varphi}|\xi^*) < \infty$. Hence $V(\xi) < \infty$.

To prove the second statement, we note that given any $M > 0$, for any $\xi \in \mathscr{K}_M$, the cost of the trajectory $\tilde{\varphi}$ constructed in the previous paragraph is bounded above by a constant depending only on M (and not on ξ). This completes the proof of the lemma. □

4.5.2 Continuity

We now establish a certain continuity property of the quasipotential V. Since V has compact level sets and the space $\mathcal{M}_1(\mathcal{Z})$ is not locally compact, we cannot expect V to be continuous on $\mathcal{M}_1(\mathcal{Z})$. In fact, for any point $\xi \in \mathcal{M}_1(\mathcal{Z})$ with $V(\xi) < \infty$, one can produce a sequence $\{\xi_n, n \geq 1\}$ such that $\xi_n \to \xi$ in $\mathcal{M}_1(\mathcal{Z})$ as $n \to \infty$, and $\langle\xi_n, \vartheta\rangle = \infty$ for all $n \geq 1$, so that $\inf_{n\geq1} V(\xi_n) = \infty$. We prove that V is continuous under the convergence of ϑ-moments when it is restricted to $\mathscr{K}$. That is, when $\xi_n, \xi \in \mathscr{K}$, $\xi_n \to \xi$ in $\mathcal{M}_1(\mathcal{Z})$, and $\langle\xi_n, \vartheta\rangle \to \langle\xi, \vartheta\rangle$ as $n \to \infty$, then $V(\xi_n) \to V(\xi)$ as $n \to \infty$. Towards this, we produce a trajectory that connects ξ to ξ_n by first moving the mass from all the large enough states z back to the state 0, then producing a constant velocity trajectory that fills the required mass from state 0 to all the large enough states z, and finally adjusting mass within a finite subset of $\mathcal{Z}$ to reach ξ_n. We show that the cost of the trajectory constructed above can be made arbitrarily small for large enough n.

Lemma 4.6. *Let $\xi_n \in \mathscr{K}$, $n \geq 1$, and $\xi \in \mathscr{K}$. Suppose that $\xi_n \to \xi$ in $\mathcal{M}_1(\mathcal{Z})$ and $\langle \xi_n, \vartheta \rangle \to \langle \xi, \vartheta \rangle$ as $n \to \infty$. Then $V(\xi_n) \to V(\xi)$ as $n \to \infty$.*

Proof. We first prove that $\limsup_{n\to\infty} V(\xi_n) \leq V(\xi)$. Fix $\varepsilon > 0$. Let $z_0 \in \mathcal{Z}$ be such that $\sum_{z>z_0} \vartheta(z)\xi(z) < \varepsilon/6$. Then choose $n_1 \geq 1$ such that $\sum_{z>z_0} \vartheta(z)\xi_n(z) < \varepsilon/3$ holds for all $n \geq n_1$; this is possible since $\xi_n \to \xi$ in $\mathcal{M}_1(\mathcal{Z})$ and $\langle \xi_n, \vartheta \rangle \to \langle \xi, \vartheta \rangle$ as $n \to \infty$. Let $t_{z_0} = 0$, $t_z = \sum_{z'=z_0+1}^{z} \xi(z')$, $z > z_0$, and $T_0 = \sum_{z'>z_0} \xi(z')$. Define the trajectory $\varphi^{(0)}$ on $[0, T_0]$ as follows. When $t \in (t_{z-1}, t_z]$ for some $z > z_0$, let

$$\dot{\varphi}_t^{(0)}(l) = \begin{cases} -1, & \text{if } l = z, \\ 1, & \text{if } l = 0, \\ 0, & \text{otherwise,} \end{cases}$$

$l \in \mathcal{Z}$, and define $\varphi_t^{(0)}(l) = \xi(l) + \int_{[0,t]} \dot{\varphi}_u^{(0)}(l)du$, $l \in \mathcal{Z}$, $t \in [0, T_1]$. Note that $\varphi_{T_0}^{(0)}(z) = \xi(z)$ for $1 \leq z \leq z_0$, $\varphi_{T_0}^{(0)}(z) = 0$ for $z > z_0$, and $\varphi_{T_0}^{(0)}(0) = \xi(0) + \sum_{z>z_0} \xi(z)$. Let $M = (\sup_{n\geq n_1} \langle \xi_n, \vartheta \rangle) \vee \langle \xi, \vartheta \rangle + 1$. Using ideas similar to those used in the proof of Lemma 4.5, it can be checked that $S_{[0,T_0]}(\varphi^{(0)}|\xi) \leq C_0(M, \overline{\lambda}, \underline{\lambda})\varepsilon$, for some constant $C_1(M, \overline{\lambda}, \underline{\lambda})$ depending on M, $\overline{\lambda}$, and $\underline{\lambda}$.

Let $\varepsilon_n = \sum_{z>z_0} \xi_n(z)$. If $\varepsilon_n > \varphi_{T_0}^{(0)}(0)$, then we move the extra mass $\varepsilon_n - \varphi_{T_0}^{(0)}(0)$ from the states $\{1, 2, \ldots, z_0\}$ to state 0 as follows. Let $T_1 = T_0 + \varepsilon_n - \varphi_{T_0}^{(0)}(0)$. When t is between $T_0 + \sum_{z'=z+1}^{z_0} \varphi_{T_0}^{(0)}(z')$ and $(T_0 + \sum_{z'=z}^{z_0} \varphi_{T_0}^{(0)}(z')) \wedge T_1$ for some $z \leq z_0$, let

$$\dot{\varphi}_t^{(1)}(l) = \begin{cases} -1, & \text{if } l = z, \\ 1, & \text{if } l = 0, \\ 0, & \text{otherwise,} \end{cases}$$

$l \in \mathcal{Z}$. Define the trajectory $\varphi^{(1)}$ on $[0, T_1]$ as follows: $\varphi_t^{(1)} = \varphi_t^{(0)}$ when $t \in [0, T_0]$; $\varphi_t^{(1)}(l) = \varphi_{T_0}^{(0)}(l) + \int_{[0,t]} \dot{\varphi}_u^{(1)}(l)du$, $l \in \mathcal{Z}$, $t \in (T_0, T_1]$. Note that $\varphi^{(1)}$ depends on n, but we suppress this in the notation for ease of readability. Again, since ε_n is smaller than $\varepsilon/3$, by using calculations similar to those used in the proof of Lemma 4.5, we see that $S_{[T_0,T_1]}(\varphi^{(1)}|\varphi_{T_0}^{(0)}) \leq C_1(M, \overline{\lambda}, \underline{\lambda})\varepsilon$ for some constant $C_1(M, \overline{\lambda}, \underline{\lambda})$ depending on M, $\overline{\lambda}$, and $\underline{\lambda}$. On the other hand, if $\varepsilon_n \leq \varphi_{T_0}^{(0)}(0)$, we set $T_1 = T_0$ and $\varphi_t^{(1)} = \varphi_t^{(0)}$ on $[0, T_1]$. In both cases, we have $\varphi_{T_1}^{(1)}(0) \geq \varepsilon_n$.

Let $T_2 = (z_0 + 1)\varepsilon_n$. We now construct another trajectory $\varphi^{(2)}$ on $[0, T_2]$ to transfer the mass ε_n from state 0 (in $\varphi_{T_1}^{(1)}$) to state $z_0 + 1$. Let $\varphi_0^{(2)} = \varphi_{T_1}^{(1)}$. When $t \in ((z-1)\varepsilon_n, z\varepsilon_n]$ for some $z \in \{1, 2, \ldots, z_0 + 1\}$, let

$$\dot{\varphi}^{(2)}_t(l) = \begin{cases} -1, & \text{if } l = z - 1, \\ 1, & \text{if } l = z, \\ 0, & \text{otherwise,} \end{cases}$$

$l \in \mathcal{Z}$, and define $\varphi^{(2)}_t(l) = \varphi^{(1)}_{T_1}(l) + \int_{[0,t]} \dot{\varphi}^{(2)}_u(l) du$, $l \in \mathcal{Z}$, $t \in (0, T_2]$. Note that $|x \log(\frac{1}{x}) - y \log(\frac{1}{y})| \le \delta + \delta \log(1/\delta)$ whenever $|x - y| \le \delta$, and that $\varepsilon_n \le \varepsilon/(z_0 \log z_0)$. Hence, using calculations similar to those done in the proof of Lemma 4.4, we see that $S_{[0,T_2]}(\varphi^{(2)} | \varphi^{(1)}_{T_1})$ can be bounded above by $C_2(M, \overline{\lambda}, \underline{\lambda}) \varepsilon \log(1/\varepsilon)$ where $C_2(M, \overline{\lambda}, \underline{\lambda})$ is a constant depending on M, $\overline{\lambda}$, and $\underline{\lambda}$, for each $n \ge n_1$ (recall that $\varphi^{(2)}$ depends on n).

Note that $\varphi^{(2)}_{T_2}(z_0 + 1) = \varepsilon_n$. We now construct a trajectory that distributes this mass ε_n from the state $z_0 + 1$ to all the states $z \ge z_0 + 1$ to match with $\xi_n(z)$. Let $t'_z = z \xi_n(z)$ for $z \ge z_0 + 2$ and $T_3 = \sum_{z \ge z_0 + 2} t'_z$. Similar to the construction in the proof of Lemma 4.4, we can now construct a trajectory $\varphi^{(3)}$ on $[0, T_3]$ such that $\varphi^{(3)}_0 = \varphi^{(2)}_{T_2}$, $\varphi^{(3)}_{T_3}(z) = \xi_n(z)$ for each $z \ge z_0 + 1$, and $S_{[0,T_3]}(\varphi^{(3)} | \varphi^{(2)}_{T_2}) \le C_3(M, \overline{\lambda}, \underline{\lambda}) \varepsilon$ for some constant $C_3(M, \overline{\lambda}, \underline{\lambda})$ depending on M, $\overline{\lambda}$, and $\underline{\lambda}$, for all $n \ge n_1$.

Finally, we construct a trajectory that connects $\varphi^{(3)}_{T_3}$ to ξ_n by adjusting the mass within the states $\{0, 1, \ldots, z_0\}$. Note that $\varphi^{(3)}_{T_3}(z) = \xi_n(z)$ for each $z \ge z_0 + 1$. Let $\mathcal{Z}_0 \subset \{1, 2, \ldots, z_0\}$ denote the set of all $z \in \{1, 2, \ldots, z_0\}$ such that $\varphi^{(3)}_{T_3}(z) > \xi_n(z)$. Similar to the construction of $\varphi^{(1)}$, for each $z \in \mathcal{Z}_0$, we move the mass $\varphi^{(3)}_{T_3}(z) - \xi_n(z)$ from state z to state 0 using unit velocity over a time duration $\varphi^{(3)}_{T_3}(z) - \xi_n(z)$. Once these mass transfers are complete, similar to the construction of $\varphi^{(2)}$, for each $z \notin \mathcal{Z}_0$, we move the mass $\xi_n(z) - \varphi^{(3)}_{T_3}(z)$ using a piecewise constant velocity trajectory from state 0 to state z over the time duration $z(\xi_n(z) - \varphi^{(3)}_{T_3}(z))$. Let $T_4 = \sum_{z \in \mathcal{Z}_0} (\varphi^{(3)}_{T_3}(z) - \xi_n(z)) + \sum_{z \notin \mathcal{Z}_0, z \le z_0} z(\xi_n(z) - \varphi^{(3)}_{T_3}(z))$. Let $\tilde{\varepsilon}_n = \max_{z \in \{1,2,\ldots,z_0\}} |\xi_n(z) - \varphi^{(3)}_{T_3}(z)|$. Then using arguments similar to those used in the proof of Lemma 4.4, we see that $S_{[0,T_4]}(\varphi^{(4)} | \varphi^{(3)}_{T_3}) \le C_4(M, \overline{\lambda}, \underline{\lambda}) \tilde{\varepsilon}_n \log(1/\tilde{\varepsilon}_n)$ for some constant $C_4(M, \overline{\lambda}, \underline{\lambda})$ depending on M, $\overline{\lambda}$, and $\underline{\lambda}$, for all $n \ge n_1$. Since $\tilde{\varepsilon}_n \to 0$ as $n \to \infty$, we can choose $n_2 \ge n_1$ such that $\tilde{\varepsilon}_n \log(1/\tilde{\varepsilon}_n) \le \varepsilon \log(1/\varepsilon)$ for all $n \ge n_2$. Therefore $S_{[0,T_4]}(\varphi^{(4)} | \varphi^{(3)}_{T_3}) \le C_4(M, \overline{\lambda}, \underline{\lambda}) \varepsilon \log(1/\varepsilon)$ for all $n \ge n_2$.

Let $T = \sum_{i=1}^4 T_i$. We now append the four paths $\varphi^{(i)}, 1 \le i \le 4$, constructed in the previous paragraphs over the time duration $[0, T]$ to get a path φ such that $\varphi_0 = \xi$, $\varphi_T = \xi_n$ and $S_{[0,T]}(\varphi | \xi) \le C(M, \overline{\lambda}, \underline{\lambda}) \varepsilon \log(1/\varepsilon)$ where $C(M, \overline{\lambda}, \underline{\lambda})$ is a constant depending on M, $\overline{\lambda}$ and $\underline{\lambda}$. Hence, for each $n \ge n_2$, we have

$$V(\xi_n) \le V(\xi) + S_{[0,T_4]}(\varphi | \xi) \le V(\xi) + C(M, \overline{\lambda}, \underline{\lambda}) \varepsilon \log(1/\varepsilon).$$

Therefore, $\limsup_{n\to\infty} V(\xi_n) \leq V(\xi) + C(M, \overline{\lambda}, \underline{\lambda})\varepsilon \log(1/\varepsilon)$. Letting $\varepsilon \to 0$ and noting that $\varepsilon \log(1/\varepsilon) \to 0$, we arrive at $\limsup_{n\to\infty} V(\xi_n) \leq V(\xi)$.

To prove $\liminf_{n\to\infty} V(\xi_n) \geq V(\xi)$, we reverse the role of ξ_n and ξ in the above argument. That is, we construct a trajectory φ on $[0, T]$ such that $\varphi_0 = \xi_n$, $\varphi_T = \xi$, and $S_{[0,T]}(\varphi|\xi_n) \leq \varepsilon_n$ for all $n \geq 1$, where $\varepsilon_n \to 0$ as $n \to \infty$. Thus, we get

$$V(\xi) \leq V(\xi_n) + \varepsilon_n.$$

Letting $n \to \infty$, we conclude that $\liminf_{n\to\infty} V(\xi_n) \geq V(\xi)$. This completes the proof of the lemma. □

Remark 4.2. The choice of n_1 in the above proof suggests that the inequality $\limsup_{n\to\infty} V(\xi_n) \leq V(\xi)$ can be proved as long as $\xi_n \to \xi$ in $\mathcal{M}_1(\mathcal{Z})$ as $n \to \infty$ and $\limsup_{n\to\infty} \langle \xi_n, \vartheta \rangle \leq \langle \xi, \vartheta \rangle$ holds. Similarly, the inequality $\liminf_{n\to\infty} V(\xi_n) \geq V(\xi)$ can be proved as long as $\xi_n \to \xi$ in $\mathcal{M}_1(\mathcal{Z})$ and $\liminf_{n\to\infty} \langle \xi_n, \vartheta \rangle \geq \langle \xi, \vartheta \rangle$ holds. This observation will be later used in the proof of the compactness of the lower level sets of V.

4.5.3 Compactness of the lower level sets of the quasipotential

Define the level sets of V by

$$\Xi(s) \coloneqq \{\xi \in \mathcal{M}_1(\mathcal{Z}) : V(\xi) \leq s\},\ s > 0.$$

In this section we establish the compactness of $\Xi(s)$ for each $s > 0$.

Lemma 4.7. *For each $s > 0$, $\Xi(s)$ is a compact subset of $\mathcal{M}_1(\mathcal{Z})$.*

Proof. We first prove an inclusion property of the level sets of V, namely, given $M > 0$ there exists $M' > 0$ such that

$$\{\xi \in \mathcal{M}_1(\mathcal{Z}) : V(\xi) \leq M\} \subset \mathscr{K}_{M'}. \tag{4.18}$$

On one hand, using Proposition 4.1 on the exponential tightness of the family $\{\wp^N, N \geq 1\}$, choose $M' > 0$ (see (4.10)) such that

$$\limsup_{N\to\infty} \frac{1}{N} \log \wp^N(\sim\mathscr{K}_{M'}) \leq -(M + 1).$$

On the other hand, using the LDP lower bound established in Lemma 4.3 and the compactness of $\mathscr{K}_{M'}$, we have

$$\liminf_{N\to\infty}\frac{1}{N}\log\wp^N(\sim\mathscr{K}_{M'})\geq -\inf_{\xi\notin\mathscr{K}_{M'}}V(\xi).$$

Combining the above two displays, we get

$$-\inf_{\xi\notin\mathscr{K}_{M'}}V(\xi)\leq\liminf_{N\to\infty}\frac{1}{N}\log\wp^N(\sim\mathscr{K}_{M'})\leq\limsup_{N\to\infty}\frac{1}{N}\log\wp^N(\sim\mathscr{K}_{M'})\leq -(M+1).$$

That is, $\xi\notin\mathscr{K}_{M'}$ implies $V(\xi)\geq M+1>M$. This shows (4.18). By Prohorov's theorem, $\mathscr{K}_M$ is a compact subset of $\mathcal{M}_1(\mathcal{Z})$; hence (4.18) shows that $\Xi(s)$ is precompact for each $s>0$.

We now show that $\Xi(s)$ is closed in $\mathcal{M}_1(\mathcal{Z})$. Let $\xi_n\in\Xi(s)$ for each $n\geq 1$ and let $\xi_n\to\xi$ in $\mathcal{M}_1(\mathcal{Z})$ as $n\to\infty$. By Fatou's lemma, we have $\liminf_{n\to\infty}\langle\xi_n,\vartheta\rangle\geq\langle\xi,\vartheta\rangle$. Hence, by Remark 4.2, we have $\liminf_{n\to\infty}V(\xi_n)\geq V(\xi)$. Thus, $\xi\in\Xi(s)$. This completes the proof of the lemma. □

4.6 The LDP upper bound

For $m\in\mathbb{N}$, define

$$\mathscr{S}_m(\Delta,M)=\{\varphi\in D([0,m],\mathcal{M}_1(\mathcal{Z})):\varphi(0)\in\mathscr{K}_M,\varphi(n)\notin K(\Delta)\text{ for all }n=1,2,\ldots,m\}.$$

That is, $\mathscr{S}_m(\Delta,M)$ denotes the set of all trajectories that start at $\mathscr{K}_M$ and do not intersect $K(\Delta)$ at all integer time points in $[0,m]$. We begin with a lemma that asserts that the elements of $\mathscr{S}_m(\Delta,M)$ for large enough m must have non-trivial cost. The key idea used in the proof comes from the compactness of level sets of the process-level large deviations rate function $S_{[0,T]}(\cdot|\nu),\nu\in K$, for any compact subset K of $\mathcal{M}_1(\mathcal{Z})$ (see Lemma 4.1).

Lemma 4.8. *For any $s>0$, $M>0$, and $\Delta>0$, there exists $m_0\in\mathbb{N}$ such that*

$$\inf\{S_{[0,m_0]}(\varphi|\varphi(0)),\varphi\in\mathscr{S}_{m_0}(\Delta,M)\}>s. \tag{4.19}$$

Proof. Suppose not. Then there exist $s>0$, $M>0$, $\Delta>0$, a sequence of positive numbers $\{\varepsilon_m,m\geq 1\}$ such that $\varepsilon_m\to 0$ as $m\to\infty$, and a sequence of trajectories $\{\varphi_m,m\geq 1\}$ such that $\varphi_m\in\mathscr{S}_m(\Delta,M)$, and $S_{[0,m]}(\varphi_m|\varphi_m(0))\leq s+\varepsilon_m$ for each $m\geq 1$.

Note that there exists an $M_1>0$ such that $\varphi_m(t)\in\mathscr{K}_{M_1}$ for each $t\in[0,m]$ and each $m\geq 1$.

Indeed, by Lemma 4.5, there exists $C_M > 0$ such that $\zeta \in K(\Delta)$ implies $V(\zeta) \leq C_M$. Thus, for each $m \geq 1$, there exist a $\bar{T}_m > 0$ and a trajectory $\bar{\varphi}_m$ on $[0, \bar{T}_m]$ such that $\bar{\varphi}_m(0) = \xi^*$, $\bar{\varphi}_m(\bar{T}_m) = \zeta \in K(\Delta)$, and $S_{[0,\bar{T}_m]}(\bar{\varphi}_m|\xi^*) \leq C_M + 1$. We extend this trajectory $\bar{\varphi}_m$ to $(\bar{T}_m, \bar{T}_m + m]$ by defining $\bar{\varphi}_m(t) = \varphi_m(t - \bar{T}_m)$ on $t \in (\bar{T}_m, \bar{T} + m]$. Note that $S_{[0,\bar{T}_m+m]}(\bar{\varphi}_m|\xi^*) \leq C_M + 1 + s + \varepsilon_m$, so that $V(\varphi_m(t)) \leq C_M + 1 + s + \varepsilon_m$ for each $t \in [0, m]$ and each $m \geq 1$. Thus, we can find an $M_1 > 0$ such that (4.18) holds with M replaced by $C_M + s + \sup_{m\geq 1} \varepsilon_m + 2$ and M' replaced by M_1. It follows that $\varphi_m(t) \in \mathscr{K}_{M_1}$ for each $t \in [0, m]$ and each $m \geq 1$.

For the above choice of M_1, using assumption (F2), choose $T_1 > 1$ such that $\mu_\zeta(t) \in K(\Delta/2)$ for each $t \geq T_1$ and each $\zeta \in \mathscr{K}_{M_1}$, where μ_ζ is the solution to the McKean-Vlasov equation (4.2) with initial condition ζ. Note that the closure of the set of all trajectories φ on $[0, T_1]$ in $D([0, T_1], \mathcal{M}_1(\mathcal{Z}))$ with initial condition $\varphi(0) \in \mathscr{K}_{M_1}$ and $\varphi(T_1) \notin K(\Delta)$ does not contain any trajectory of the McKean-Vlasov equation (4.2). It follows from Lemma 4.1 that

$$\beta \coloneqq \inf\{S_{[0,T_1]}(\varphi|\varphi(0)), \varphi(0) \in \mathscr{K}_{M_1}, \varphi(n) \notin K(\Delta) \text{ for each } n = 1, 2, \ldots, \lfloor T_1 \rfloor\} > 0.$$

Therefore, noting that $\varphi_m(t) \in \mathscr{K}_{M_1}$ for each $t \in [0, m]$ and $m \geq 1$, we see that

$$\begin{aligned} S_{[0,m]}(\varphi_m|\varphi_m(0)) &\geq \sum_{n=1}^{\lfloor m/T_1 \rfloor} S_{[(n-1)T_1, nT_1]}(\varphi_m|\varphi_m((n-1)T)) \\ &\geq \left\lfloor \frac{m}{T_1} \right\rfloor \beta \\ &\to \infty \text{ as } m \to \infty, \end{aligned}$$

which contradicts our assumption. This completes the proof of the lemma. □

With a slight abuse of notation, given $A \subset \mathcal{M}_1(\mathcal{Z})$, $s > 0$, and $T > 0$, define

$$\Phi_A^{[0,T]}(s) \coloneqq \{\varphi \in D([0, T], \mathcal{M}_1(\mathcal{Z})) : \varphi(0) \in A, S_{[0,T]}(\varphi|\varphi(0)) \leq s\}.$$

We now prove a certain containment property for elements of $\mathcal{M}_1(\mathcal{Z})$ that can arise as end-points of trajectories in $\Phi_{K(\Delta)}^{[0,T]}(s)$, $s > 0$ and $\Delta > 0$, i.e., points $\xi \in \mathcal{M}_1(\mathcal{Z})$ such that there exists a trajectory φ with $\varphi_0 \in K(\Delta)$ and $S_{[0,T]}(\varphi|\varphi_0) \leq s$. We prove that such points are not far from the lower level sets of V in $\mathcal{M}_1(\mathcal{Z})$. This connection between trajectories over finite time horizons and the level sets of the quasipotential V is the key to transfer the process-level LDP upper bound in Theorem 4.2 to the LDP upper bound for the family of invariant measures $\{\wp^N, N \geq 1\}$.

Lemma 4.9. *For any $s > 0$ and $\delta > 0$ there exists $\Delta > 0$ and $T_1 \geq 1$ such that for all $T \geq T_1$,*

$$\{\varphi(T) : \varphi \in \Phi^{[0,T]}_{K(\Delta)}(s)\} \subset \{\xi \in \mathcal{M}_1(\mathscr{Z}) : d(\xi, \Xi(s)) \leq \delta\}. \tag{4.20}$$

Proof. Suppose not. Then there exist $s > 0$, $\delta > 0$, sequences $\{\Delta_n, n \geq 1\}$, $\{T_n, n \geq 1\}$ such that $\Delta_n \downarrow 0$ and $T_n \uparrow \infty$ as $n \to \infty$, and trajectories $\varphi_n \in \Phi^{[0,T_n]}_{K(\Delta_n)}(s)$ such that $d(\varphi_n(T_n), \Xi(s)) > \delta$ for each $n \geq 1$. Let $\xi_n = \varphi_n(T_n)$, $n \geq 1$. By Lemma 4.6, there exists a $T' > 0$ and a sequence $\{\varepsilon_n, n \geq 1\}$, with $\varepsilon_n \to 0$ as $n \to \infty$, such that for any $\zeta' \in K(\Delta_n)$ there exists a trajectory $\bar{\varphi}^{\zeta'}$ on $[0, T']$ such that $\bar{\varphi}^{\zeta'}(0) = \xi^*, \bar{\varphi}^{\zeta'}(T') = \zeta'$, and $S_{[0,T']}(\bar{\varphi}^{\zeta'}|\xi^*) \leq \varepsilon_n$. For each $n \geq 1$, let $\tilde{\varphi}_n$ be the trajectory on $[0, T' + T_n]$ defined as follows. Let $\tilde{\varphi}_n(0) = \xi^*$; $\tilde{\varphi}_n(t) = \bar{\varphi}^{\varphi_n(0)}(t)$ on $t \in [0, T']$; $\tilde{\varphi}_n(t) = \varphi_n(t - T')$ on $t \in (T', T' + T_n]$. In particular, $\tilde{\varphi}_n(T' + T_n) = \xi_n$. Clearly, $S_{[0,T'+T_n]}(\tilde{\varphi}_n|\xi^*) \leq s + \varepsilon_n$. It follows that $V(\xi_n) \leq s + \varepsilon_n$. Using the compactness of the lower level sets of V (see Lemma 4.7), we can find a convergent subsequence of $\{\xi_n, n \geq 1\}$; after re-indexing and denoting this convergent subsequence by $\{\xi_n, n \geq 1\}$, let $\xi_n \to \xi$ in $\mathcal{M}_1(\mathscr{Z})$ as $n \to \infty$. By assumption, $d(\xi_n, \Xi(s)) > \delta$ for each $n \geq 1$, and hence $d(\xi, \Xi(s)) \geq \delta$. Using the lower semicontinuity of V, we see that

$$V(\xi) \leq \liminf_{n\to\infty} V(\xi_n) \leq \liminf_{n\to\infty}(s + \varepsilon_n) = s.$$

Hence $\xi \in \Xi(s)$. This contradicts $d(\xi, \Xi(s)) \geq \delta$, which is a consequence of our assumption. This proves the lemma. □

We are now ready to prove the LDP upper bound for the family $\{\wp^N, N \geq 1\}$. The proof relies on the uniform LDP upper bound in Theorem 4.2, the exponential tightness of the family $\{\wp^N, N \geq 1\}$, the containment property established in Lemma 4.9, an estimate on the probability that μ^N lies in $\mathscr{S}_m(M, \Delta)$ (which uses the process-level uniform LDP upper bound in Theorem 4.2 and the result of Lemma 4.8), and finally the strong Markov property of μ^N.

Lemma 4.10. *For any $\gamma > 0$, $\delta > 0$, and $s > 0$, there exists $N_0 \geq 1$ such that*

$$\wp^N\{\zeta \in \mathcal{M}_1(\mathscr{Z}) : d(\zeta, \Xi(s)) \geq \delta\} \leq \exp\{-N(s - \gamma)\}$$

for all $N \geq N_0$.

Proof. Fix $\gamma > 0$, $\delta > 0$, and $s > 0$. Choose $M > 0$ and $N_1 \geq 1$ such that $\wp^N(\sim\mathscr{K}_M) \leq \exp\{-Ns\}$ for all $N \geq N_1$; this is possible from the exponential tightness of the family $\{\wp^N, N \geq 1\}$, see Proposition 4.1. For the given $s > 0$ and $\delta > 0$, from Lemma 4.9, choose $\Delta > 0$ and $T_1 > 0$ such that (4.20) holds for all $T \geq T_1$. For the above choice of $\Delta > 0$ and $M > 0$, by

Lemma 4.8, choose $m_0 \in \mathbb{N}$ such that such that (4.19) holds. By (4.19) and the compactness of $\Phi^{[0,m_0]}_{\mathscr{K}_M}$ in $D([0,m_0],\mathcal{M}_1(\mathcal{Z}))$, the closure of $\mathscr{S}_{m_0}(\Delta, M)$ does not intersect $\Phi^{[0,m_0]}_{\mathscr{K}_M}$. It follows that there exists a $\delta_0 > 0$ such that $\varphi \in \mathscr{S}_{m_0}(\Delta, M)$ implies $\rho(\varphi, \Phi^{[0,m_0]}_{\mathscr{K}_M}(s)) \geq \delta_0$. Hence by the uniform LDP upper bound in Theorem 4.2, there exists $N_2 \geq N_1$ such that

$$\begin{aligned}
\mathbb{P}^N_\zeta(\mu^N \in \mathscr{S}_{m_0}(\Delta, M)) &\leq \mathbb{P}^N_\zeta(\rho(\mu^N, \Phi^{[0,m_0]}_{\mathscr{K}_M}) \geq \delta_0) \\
&\leq \exp\{-N(s-\gamma/2)\}
\end{aligned} \tag{4.21}$$

for all $\zeta \in \mathscr{K}_M \cap \mathcal{M}^N_1(\mathcal{Z})$ and $N \geq N_2$. Thus, with $T = m_0 + T_1$ and $N \geq N_2$, we have

$$\begin{aligned}
&\wp^N\{\zeta \in \mathcal{M}_1(\mathcal{Z}) : d(\zeta, \Xi(s)) \geq \delta\} \\
&\quad = \int_{\mathcal{M}^N_1(\mathcal{Z})} \mathbb{P}^N_\zeta(d(\mu^N(T), \Xi(s)) \geq \delta)\wp^N(d\zeta) \\
&\quad \leq \exp\{-Ns\} + \int_{\mathscr{K}_M \cap \mathcal{M}^N_1(\mathcal{Z})} \mathbb{P}^N_\zeta(d(\mu^N(T), \Xi(s)) \geq \delta)\wp^N(d\zeta) \\
&\quad \leq \exp\{-Ns\} + \sup_{\zeta \in \mathscr{K}_M \cap \mathcal{M}^N_1(\mathcal{Z})} \mathbb{P}^N_\zeta(\mu^N \in \mathscr{S}_{m_0}(\Delta, M)) \\
&\qquad + \int_{\mathscr{K}_M \cap \mathcal{M}^N_1(\mathcal{Z})} \mathbb{P}^N_\zeta(\mu^N \notin \mathscr{S}_{m_0}(\Delta, M), d(\mu^N(T), \Xi(s)) \geq \delta)\wp^N(d\zeta) \\
&\quad \leq \exp\{-Ns\} + \exp\{-N(s-\gamma/2)\} \\
&\qquad + \int_{\mathscr{K}_M \cap \mathcal{M}^N_1(\mathcal{Z})} \mathbb{P}^N_\zeta(\mu^N \notin \mathscr{S}_{m_0}(\Delta, M), d(\mu^N(T), \Xi(s)) \geq \delta)\wp^N(d\zeta);
\end{aligned} \tag{4.22}$$

here the first equality follows since $\wp^N$ is invariant to time shifts, the first inequality follows from the choice of M, and the third inequality follows from (4.21).

To bound the integrand in the third term above, let $T' \geq T_1$ and $\zeta' \in K(\Delta)$. Choose $0 < \delta' < \delta$ (depending on T and s, and not on ζ' and T') such that $\rho(\varphi_1, \varphi_2) < \delta'/2$ implies $d(\varphi_1(T'), \varphi_2(T')) < \delta/2$ whenever $\varphi_1 \in D([0,T'], \mathcal{M}_1(\mathcal{Z}))$ and $\varphi_2 \in \Phi^{[0,T']}_{\zeta'}$. Note that if a trajectory φ on $[0,T']$ with initial condition $\varphi(0) = \zeta'$ is such that $\rho(\varphi, \Phi^{[0,T']}_{\zeta'}(s)) < \delta'/2$, then there exists a trajectory $\varphi' \in \Phi^{[0,T']}_{\zeta'}(s)$ such that $\rho(\varphi, \varphi') < \delta'/2$. By the choice of δ', we have $d(\varphi(T'), \varphi'(T')) < \delta/2$. By Lemma 4.9, we find that $d(\varphi'(T'), \Xi(s)) \leq \delta'/2$. Hence by triangle inequality $d(\varphi(T'), \Xi(s)) < \delta/2 + \delta'/2 < \delta$. The contrapositive of the above statement is

$$d(\varphi(T'), \Xi(s)) \geq \delta \Rightarrow \rho(\varphi, \Phi^{[0,T']}_{\zeta'}(s)) \geq \delta'/2.$$

We therefore conclude that

$$\mathbb{P}^N_{\zeta'}(d(\mu^N(T'),\Xi(s)) \geq \delta) \leq \mathbb{P}^N_{\zeta'}(\rho(\mu^N, \Phi^{[0,T']}_{\zeta'}(s)) \geq \delta'/2) \tag{4.23}$$

for all $T' \geq T_1$, $\zeta' \in \mathscr{K}(\Delta) \cap \mathcal{M}_1^N(\mathcal{Z})$, and $N \geq 1$.

Note that the integrand in the last term of (4.22) can be upper bounded by

$$\begin{aligned}
&\mathbb{P}^N_{\zeta}(\mu^N \notin \mathscr{S}_{m_0}(\Delta, M), d(\mu^N(T),\Xi(s)) \geq \delta) \\
&\quad = \mathbb{P}^N_{\zeta}(\mu^N(m) \in K(\Delta) \text{ for some } m = 1,2,\ldots,m_0,\ d(\mu^N(T),\Xi(s)) \geq \delta) \\
&\quad \leq \sum_{m=1}^{m_0} \sup_{\zeta' \in K(\Delta)\cap\mathcal{M}_1^N(\mathcal{Z})} \mathbb{P}^N_{\zeta'}(d(\mu^N(T-m),\Xi(s)) \geq \delta) \\
&\quad \leq \sum_{m=1}^{m_0} \sup_{\zeta' \in K(\Delta)\cap\mathcal{M}_1^N(\mathcal{Z})} \mathbb{P}^N_{\zeta'}(\rho((\mu^N(t), t \in [0,T-m]), \Phi^{[0,T-m]}_{\zeta'}(s)) \geq \delta'/2)
\end{aligned} \tag{4.24}$$

where the first inequality follows from the strong Markov property of μ^N and the second inequality follows from (4.23) by the choice of T. By the uniform LDP upper bound in Theorem 4.2, for each $m = 1,2,\ldots m_0$, there exist $N(m) \geq N_2$ such that

$$\mathbb{P}^N_{\zeta'}(\rho((\mu^N(t), t \in [0,T-m]), \Phi^{[0,T-m]}_{\zeta'}(s)) \geq \delta'/2) \leq \exp\{-N(s-\gamma/2)\}$$

for all $\zeta' \in \mathscr{K}(\Delta) \cap \mathcal{M}_1^N(\mathcal{Z})$ and $N \geq N(m)$. Put $N_3 = \max\{N(m), m = 1,2,\ldots,m_0, N_1, N_2\}$. Then (4.24) yields

$$\mathbb{P}^N_{\zeta}(\mu^N \notin \mathscr{S}_{m_0}(\Delta, M), d(\mu^N(T),\Xi(s)) \geq \delta) \leq m_0 \exp\{-N(s-\gamma/2)\}$$

for all $\zeta \in \mathscr{K}_M \cap \mathcal{M}_1^N(\mathcal{Z})$ and $N \geq N_3$. Substitution of this back in (4.22) yields

$$\wp^N\{\zeta \in \mathcal{M}_1(\mathcal{Z}) : d(\zeta,\Xi(s)) \geq \delta\} \leq \exp\{-Ns\} + (m_0+1)\exp\{-N(s-\gamma/2)\}$$

for all $N \geq N_3$. Finally, choose $N_0 \geq N_3$ such that $1 + (m_0+1)\exp\{N\gamma/2\} \leq \exp\{N\gamma\}$ for all $N \geq N_0$. Then the above display becomes

$$\wp^N\{\zeta \in \mathcal{M}_1(\mathcal{Z}) : d(\zeta,\Xi(s)) \geq \delta\} \leq \exp\{-N(s-\gamma)\}$$

for all $N \geq N_0$. This completes the proof of the lemma. □

4.7 Proof of Theorem 4.1

We now complete the proof of Theorem 4.1.

- (Compactness of level sets). For any $s > 0$, by Lemma 4.7, the set $\Xi(s) = \{\xi \in \mathcal{M}_1(\mathcal{Z}) : V(\xi) \leq s\}$ is a compact subset of $\mathcal{M}_1(\mathcal{Z})$;
- (LDP lower bound). Given $\gamma > 0$, $\delta > 0$, and $\xi \in \mathcal{M}_1(\mathcal{Z})$, by Lemma 4.3, there exists $N_0 \geq 1$ such that

$$\wp^N\{\zeta \in \mathcal{M}_1(\mathcal{Z}) : d(\zeta, \xi) < \delta\} \geq \exp\{-N(V(\xi) + \gamma)\}$$

  for all $N \geq N_0$;
- (LDP upper bound). Given $\gamma > 0$, $\delta > 0$, and $s > 0$, by Lemma 4.10, there exists $N_0 \geq 1$ such that

$$\wp^N\{\zeta \in \mathcal{M}_1(\mathcal{Z}) : d(\zeta, \Xi(s)) \geq \delta\} \leq \exp\{-N(s - \gamma)\}$$

  for all $N \geq N_0$.

This completes the proof of Theorem 4.1.

4.8 Two counterexamples

In this section, for two non-interacting counterexamples described in Section 4.1.2, we prove that the quasipotential is not equal to the relative entropy with respect to the corresponding globally asymptotically stable equilibrium. These two counterexamples are (i) a system of non-interacting M/M/1 queues, and (ii) a system of non-interacting nodes in a wireless local area network (WLAN) with constant forward transition rates. We detail the proofs in the case of non-interacting M/M/1 queues. Similar arguments carry over to the case of non-interacting WLAN system with constant forward transition rates as well.

4.8.1 A system of non-interacting M/M/1 queues

Recall the system of non-interacting M/M/1 queues described in Section 4.1.2.1. Recall the relative entropy from (4.4) and the process-level large deviations rate function from (4.7). Also recall the function ϑ and the compact sets $\mathscr{K}_M$, $M > 0$. Define the quasipotential

$$V_Q(\xi) := \inf\{S^Q_{[0,T]}(\varphi|\xi^*_Q), \varphi(0) = \xi^*_Q, \varphi(T) = \xi, T > 0\}, \xi \in \mathcal{M}_1(\mathcal{Z}),$$

where S^Q is defined by (4.7) with $\mathcal{E}$ replaced by $\mathcal{E}_Q$ and L_ζ replaced by L^Q for each $\zeta \in \mathcal{M}_1(\mathcal{Z})$.

We first prove that the quasipotential V_Q is not finite outside $\mathscr{K}$. The key property used for this is the fact that the attractor ξ^*_Q has geometric decay. As a consequence $\langle \xi^*_Q, \vartheta\rangle < \infty$. Using this property, we first show that if $\xi \notin \mathscr{K}$, then the associated quasipotential evaluated at ξ cannot be finite. This is shown by producing a lower bound for the cost of any trajectory starting at ξ^*_Q and ending at $\xi \notin \mathscr{K}$ from the rate function in (4.7).

Lemma 4.11. *If $\xi \in \mathcal{M}_1(\mathcal{Z})$ is such that $\xi \notin \mathscr{K}$, then $V_Q(\xi) = \infty$.*

Proof. Fix $\xi \in \mathcal{M}_1(\mathcal{Z})$. Let $T > 0$ and $\varphi \in D([0,T], \mathcal{M}_1(\mathcal{Z}))$ be such that $\varphi_0 = \xi^*_Q$ and $\varphi_T = \xi$. For each $n \geq 1$, define f_n by

$$f_n(z) = \begin{cases} z, \text{ if } z \leq n \\ 2n - z, \text{ if } n + 1 \leq z \leq 2n, \\ 0, \text{ if } z > 2n, \end{cases}$$

and define $f_\infty(z) = z$ for each $z \in \mathcal{Z}$. We first assume that $\langle \xi, f_\infty\rangle = \infty$. In particular, $\xi \notin \mathscr{K}$. Using the function f_n in place of f in the RHS of (4.7), we have

$$\begin{aligned} S^Q_{[0,T]}(\varphi|\xi^*_Q) &\geq \langle \varphi_T, f_n\rangle - \langle \xi^*_Q, f_n\rangle - \int_{[0,T]} \langle \varphi_u, L^Q f_n\rangle - \int_{[0,T]} \sum_{(z,z')\in\mathcal{E}_Q} \tau(f_n(z') - f_n(z))\lambda_{z,z'}\varphi_u(z)du \\ &= \langle \varphi_T, f_n\rangle - \langle \xi^*_Q, f_n\rangle - \int_{[0,T]} \sum_{(z,z')\in\mathcal{E}_Q} (\exp\{f_n(z') - f_n(z)\} - 1)\lambda_{z,z'}\varphi_u(z)du, \end{aligned}$$

where $\lambda_{z,z+1} = \lambda_f$, $z \in \mathcal{Z}$, and $\lambda_{z,z-1} = \lambda_b$, $z \in \mathcal{Z} \setminus \{0\}$. Noting that $f_n(z') - f_n(z)$ is either 1, 0 or -1 for each $(z, z') \in \mathcal{E}_Q$, we have $\sum_{(z,z')\in\mathcal{E}_Q}(\exp\{f_n(z') - f_n(z)\} - 1)\lambda_{z,z'}\varphi_u(z) \leq 2(e - 1)\lambda_b$ for each $u \in [0,T]$. Hence the above becomes

$$S^Q_{[0,T]}(\varphi|\xi^*_Q) \geq \langle \varphi_T, f_n\rangle - \langle \xi^*_Q, f_n\rangle - 2(e - 1)\lambda_b T.$$

Note that $\langle \xi^*_Q, f_\infty\rangle < \infty$. Hence, letting $n \to \infty$ and using the monotone convergence theorem, we conclude that $S^Q_{[0,T]}(\varphi|\xi^*_Q) = \infty$.

We now assume that $\xi \notin \mathscr{K}$ is such that $\langle \xi, f_\infty\rangle < \infty$. Let $T > 0$ and $\varphi \in D([0,T], \mathcal{M}_1(\mathcal{Z}))$ be such that $\varphi_0 = \xi^*_Q$ and $\varphi_T = \xi$. Without loss of generality, we can assume that $\sup_{t\in[0,T]}\langle \varphi_t, f_\infty\rangle <$

∞; otherwise the argument in the above paragraph shows that $S^Q_{[0,T]}(\varphi|\xi^*_Q) = \infty$. Define

$$\vartheta_n(z) = \begin{cases} \vartheta(z), \text{ if } z \le n, \\ \vartheta(2n - z) \text{ if } n + 1 \le z \le 2n, \\ 0, \text{ if } z > 2n. \end{cases}$$

Using ϑ_n in the RHS of (4.7), we get

$$S^Q_{[0,T]}(\varphi|\xi^*_Q) \ge \langle \xi, \vartheta_n\rangle - \langle \xi^*_Q, \vartheta_n\rangle - \int_{[0,T]} \sum_{(z,z')\in\mathcal{E}_Q} (\exp\{\vartheta_n(z') - \vartheta_n(z)\} - 1)\lambda_{z,z'}\varphi_u(z)du.$$

Noting that $\vartheta_n(z') - \vartheta_n(z)$ can be upper bounded by $1 + \log(z + 1)$ for each $(z, z') \in \mathcal{E}_Q$, it follows that $\sum_{(z,z')\in\mathcal{E}_Q}(\exp\{\vartheta_n(z') - \vartheta_n(z)\} - 1)\lambda_{z,z'}\varphi_u(z) \le 2\lambda_b(e(\sup_{t\in[0,T]}\langle\varphi_t, f_\infty\rangle + 1) - 1)$ for each $u \in [0, T]$. Hence the above display becomes

$$S^Q_{[0,T]}(\varphi|\xi^*_Q) \ge \langle \xi, \vartheta_n\rangle - \langle \xi^*_Q, \vartheta_n\rangle - 2\lambda_b(e(\sup_{t\in[0,T]}\langle\varphi_t, f_\infty\rangle + 1) - 1)T.$$

As before, letting $n \to \infty$, using the monotone convergence theorem, and noting that $\xi^*_Q \in \mathscr{K}$, we conclude that $S^Q_{[0,T]}(\varphi|\xi^*_Q) = \infty$.

Since $\xi \notin \mathscr{K}$, $T > 0$, and $\varphi \in D([0, T], \mathcal{M}_1(\mathcal{Z}))$ such that $\varphi_0 = \xi^*_Q$ and $\varphi_T = \xi$ are arbitrary, the proof of the lemma is complete. □

We now prove the main result of this section, namely, the quasipotential V_Q is not equal to the relative entropy $I(\cdot\|\xi^*_Q)$.

Proposition 4.2. *Let $\xi \in \mathcal{M}_1(\mathcal{Z})$ be such that $\langle \xi, f_\infty\rangle < \infty$ and $\xi \notin \mathscr{K}$. Then $I(\xi\|\xi^*_Q) < \infty$ and $V(\xi) = \infty$. In particular, $V \ne I(\cdot\|\xi^*_Q)$.*

Proof. By the Donsker-Varadhan variational formula (see Donsker and Varadhan [31, Lemma 2.1]), for any $\xi \in \mathcal{M}_1(\mathcal{Z})$ and any bounded function f on $\mathcal{Z}$, we have

$$I(\xi\|\xi^*_Q) \ge \langle \xi, f\rangle - \log\left(\sum_{z\in\mathcal{Z}} \exp\{f(z)\}\xi^*_Q(z)\right).$$

Recall the definition of f_n and f_∞ from the proof of Lemma 4.11. Let $\bar\beta > 0$ be such that $\sum_{z\in\mathcal{Z}} \exp\{\bar\beta z\}\xi^*_Q(z) < \infty$. Replacing f by $\bar\beta f_n$ in the above display, letting $n \to \infty$ and using the monotone convergence theorem, we arrive at

$$\bar{\beta}\langle\xi, f_\infty\rangle \leq I(\xi\|\xi_Q^*) + \log\left(\sum_{z\in\mathcal{Z}} \exp\{\bar{\beta}z\}\xi_Q^*(z)\right).$$

It follows that

$$\{\xi \in \mathcal{M}_1(\mathcal{Z}) : I(\xi\|\xi_Q^*) < \infty\} \subset \{\xi \in \mathcal{M}_1(\mathcal{Z}) : \langle\xi, f_\infty\rangle < \infty\}.$$

On the other hand, since $\langle\xi_Q^*, f_\infty\rangle < \infty$, it is easy to check that $\{\xi \in \mathcal{M}_1(\mathcal{Z}) : I(\xi\|\xi_Q^*) < \infty\} \supset \{\xi \in \mathcal{M}_1(\mathcal{Z}) : \langle\xi, f_\infty\rangle < \infty\}$.

Let $\xi \in \mathcal{M}_1(\mathcal{Z})$ be such that $\langle\xi, \vartheta\rangle = \infty$ and $\langle\xi, f_\infty\rangle < \infty$. Then the above yields $I(\xi\|\xi_Q^*) < \infty$. By Lemma 4.11, we see that $V_Q(\xi) = \infty$. This completes the proof of the proposition. □

4.8.2 A non-interacting WLAN system with constant forward rates

Recall the model described in Section 4.1.2.2. Define the quasipotential

$$V_W(\xi) := \inf\{S_{[0,T]}^W(\varphi|\xi_W^*), \varphi_0 = \xi_W^*, \varphi_T = \xi, T > 0\},\ \xi \in \mathcal{M}_1(\mathcal{Z}),$$

where S^W is defined by (4.7) with $\mathcal{E}$ replaced by $\mathcal{E}_W$ and L_ζ replaced by L^W for each $\zeta \in \mathcal{M}_1(\mathcal{Z})$. We now state the main result for this non-interacting wireless local area network.

Proposition 4.3. *Let $\xi \in \mathcal{M}_1(\mathcal{Z})$ be such that $\langle\xi, f_\infty\rangle < \infty$ and $\xi \notin \mathscr{K}$. Then $I(\xi\|\xi_W^*) < \infty$ and $V(\xi) = \infty$. In particular, $V_W \neq I(\cdot\|\xi_W^*)$.*

We start with the following lemma. The proof follows along similar lines of the proof of Lemma 4.11 by noting that $\langle\xi_W^*, \vartheta\rangle < \infty$, and it is left to the reader.

Lemma 4.12. *If $\xi \in \mathcal{M}_1(\mathcal{Z})$ is such that $\xi \notin \mathscr{K}$, then $V_W(\xi) = \infty$.*

Using the above lemma, we can now prove Proposition 4.3 along similar lines of the proof of Proposition 4.2 in the previous section.

4.A Proofs of Section 4.2

Proof of Lemma 4.1. Fix $T > 0$, $s > 0$, and $K \subset \mathcal{M}_1(\mathcal{Z})$ compact. Given $\nu \in K$, $\varphi \in \Phi_\nu^{[0,T]}(s)$ and a finite set $B \subset \mathcal{Z}$, choosing $f(t, z) = \mathbf{1}_{\{z \in B\}}$, (4.6) yields

$$\varphi_t(B) - \varphi_r(B) = \int_{[s,t]} \sum_{(z,z') \in \mathcal{E}} (f(z') - f(z))(1 + h_\varphi(u, z, z'))\lambda_{z,z'}(\varphi_u)\varphi_u(z)du$$

for all $0 \leq r < t \leq T$. Noting that $h_\varphi \geq -1$, we get

$$|\varphi_t(B) - \varphi_r(B)| \leq \int_{[0,T]} \sum_{(z,z') \in \mathcal{E}} (1 + h_\varphi(u, z, z')) \times \mathbf{1}_{\{u \in [r,t]\}} \lambda_{z,z'}(\varphi_u)\varphi_u(z)du. \tag{4.25}$$

Noting that

$$\sup \left\{ \int_{[0,T]} \sum_{(z,z') \in \mathcal{E}} \tau^*(h_\varphi(u, z, z'))\lambda_{z,z'}(\varphi_u)\varphi_u(z)du, \varphi \in \Phi_\nu^{[0,T]}(s), \nu \in K \right\} \leq s,$$

it follows that the family $\{1 + h_\varphi, \varphi \in \Phi_\nu^{[0,T]}(s), \nu \in K\}$ is uniformly integrable. That is,

$$\sup \left\{ \int_{[0,T]} (1 + h_\varphi(u, z, z')) \times \mathbf{1}_{\{1+h_\varphi \geq M\}} \lambda_{z,z'}(\varphi_u)\varphi_u(z)du, \varphi \in \Phi_\nu^{[0,T]}, \nu \in K \right\} \to 0$$

as $M \to \infty$. Hence for any $M > 0$, using the boundedness of the transition rates (from assumption (E2)), (4.25) yields

$$\begin{aligned}
&|\varphi_t(B) - \varphi_r(B)| \\
&\leq 2M\overline{\lambda}(t - r) + \int_{[0,T]} \sum_{(z,z') \in \mathcal{E}} (1 + h_\varphi(u, z, z')) \times \mathbf{1}_{\{1+h_\varphi \geq M\}} \lambda_{z,z'}(\varphi_u)\varphi_u(z)du.
\end{aligned}$$

for all $0 \leq r < t \leq T$, and $B \subset \mathcal{M}_1(\mathcal{Z})$. It follows that

$$\begin{aligned}
&\sup_{\varphi \in \cup_{\nu \in K} \Phi_\nu^{[0,T]}(s)} \sup_{t,r:|t-r| \leq \delta} d(\varphi_t, \varphi_r) \\
&\quad \leq 2M\overline{\lambda}\delta + \sup_{\varphi \in \cup_{\nu \in K} \Phi_\nu^{[0,T]}(s)} \sup_{t,r:|t-r| \leq \delta} \int_{[0,T]} \sum_{(z,z') \in \mathcal{E}} (1 + h_\varphi(u, z, z')) \\
&\qquad \times \mathbf{1}_{\{1+h_\varphi \geq M\}} \lambda_{z,z'}(\varphi_u)\varphi_u(z)du
\end{aligned}$$

Letting $\delta \to 0$ first and then $M \to \infty$, we arrive at

$$\lim_{\delta \downarrow 0} \sup_{\varphi \in \cup_{\nu \in K} \Phi_\nu^{[0,T]}(s)} \sup_{t,r:|t-r|\le\delta} d(\varphi_t, \varphi_r) = 0.$$

Hence it follows that $\cup_{\nu\in K}\Phi_\nu^{[0,T]}(s)$ is precompact in $D([0,T],\mathcal{M}_1(\mathcal{Z}))$ (see, for example, Billingsley [12, Theorem 12.3]).

To show that $\cup_{\nu\in K}\Phi_\nu^{[0,T]}(s)$ is closed, let $\{\varphi_n, n \ge 1\} \subset \cup_{\nu\in K}\Phi_\nu^{[0,T]}(s)$ and suppose that $\varphi_n \to \varphi$ in $D([0,T],\mathcal{M}_1(\mathcal{Z}))$. Note that the mapping

$$\mathcal{M}_1(\mathcal{Z}) \times \mathbb{R}^\infty \ni (u,v) \mapsto \sup_{\alpha\in\mathbb{R}^\infty} \left\{ \langle \alpha, v - \Lambda_u^* u\rangle - \sum_{(z,z')\in\mathcal{E}} \tau(\alpha(z') - \alpha(z))\lambda_{z,z'}(u)u(z) \right\}$$

is lower semicontinuous (see, for example, Berge [7, Theorem 1, page 115]). Hence, by Fatou's lemma, we see that

$$S_{[0,T]}(\varphi|\varphi(0)) \le \liminf_{n\to\infty} S_{[0,T]}(\varphi_n|\varphi_n(0)) \le s,$$

and it follows that $\cup_{\nu\in K}\Phi_\nu^{[0,T]}(s)$ is closed. Consequently, $\cup_{\nu\in K}\Phi_\nu^{[0,T]}(s)$ is a compact subset of $D([0,T],\mathcal{M}_1(\mathcal{Z}))$. □

Proof of Theorem 4.2. Let $\mathcal{M}_1^N(\mathcal{Z}) \ni \nu_N \to \nu$ in $\mathcal{M}_1(\mathcal{Z})$ as $N \to \infty$. Then the family $\{\mu_{\nu_N}^N, N \ge 1\}$ satisfies the LDP on $D([0,T],\mathcal{M}_1(\mathcal{Z}))$ with rate function $S_{[0,T]}(\cdot|\nu)$ (see Léonard [55, Theorem 3.1] and Borkar and Sundaresan [15, Theorem 3.2]). By Lemma 4.1 on the compactness of level sets of $S_{[0,T]}(\cdot|\nu)$, $\nu \in K$, for any compact subset K of $\mathcal{M}_1(\mathcal{Z})$, it follows that the family $\{\mu_\nu^N, \nu \in \mathcal{M}_1^N(\mathcal{Z}), N \ge 1\}$ satisfies the uniform large deviation principle on $D([0,T],\mathcal{M}_1(\mathcal{Z}))$ over the class of compact subsets of $\mathcal{M}_1(\mathcal{Z})$ with the family of rate functions $\{S_{[0,T]}(\cdot|\nu), \nu \in \mathcal{M}_1(\mathcal{Z})\}$ (see Budhiraja and Dupuis [17, Propositions 1.12, 1.14]). □

Chapter 5

Conclusion

This thesis studied the large time behaviour, metastability, and the asymptotics of the invariant measure in models of Markovian mean-field interacting particle systems. The general principle in this thesis has been to first study the process-level large deviations of the underlying stochastic model and then use this to study the large time behaviour, metastability, and the large deviations of the invariant measure. In Chapter 2, we used the existing results on the process-level large deviations of finite-state mean-field models to study its large time behaviour and metastability. In Chapter 3, we proved the process-level large deviations of two time scale mean-field models and then used the results of Chapter 2 to study its large time behaviour and metastability. In Chapter 4, we considered countable-state mean-field models, extended the process-level LDP to the uniform LDP over the class of compact subsets, and then used this to prove the large deviations of the invariant measure.

This general principle of studying the large time behaviour of stochastic systems by first studying the large deviations over finite time horizons and then passing to the large time limit has been employed in other contexts as well [80, 23, 15, 58]. However, in Chapter 4, we demonstrated two counterexamples where the LDP for the family of invariant measures holds but the rate function for this LDP is not governed by the usual Freidlin-Wentzell quasipotential. This suggests that, to apply the above general principle, the underlying stochastic dynamics must have some "good" properties, especially for infinite-dimensional problems where the state space of the underlying stochastic process is not locally compact. In the case of finite-dimensional problems studied in Chapter 2 and Chapter 3, the assumptions imposed on the particle transition rates sufficed to prove the uniform LDP for the underlying model and the necessary small-cost connection properties, which enabled us to carry out this general procedure. In Chapter 4, however, we needed the additional $1/z$-decay of the forward transition rates to carry out the procedure.

5.1 Future directions

We now discuss some open questions and future directions.

Uniform large deviations over the class of open subsets for countable-state mean-field models: To study the large deviations of the family of invariant measures of countable-state mean-field models in Chapter 4, we used the uniform large deviations of the empirical measure process over the class of *compact subsets* of the space of probability measures on $\mathcal{Z}$. An interesting open question here is to establish the uniform LDP for the empirical measure process over the class of *open subsets* of the space of probability measures on $\mathcal{Z}$. In the case of finite-state models, since the space of probability measures on a finite set is locally compact, the former uniform LDP over the class of compact subsets of the space implies the uniform LDP over the class of open subsets. This is no longer true for countable-state models since the closure of open sets are not necessarily compact. It is not clear if the countable state space model studied in Chapter 4 satisfies the uniform LDP over the class of open subsets of the space; further assumptions on the model are perhaps required to established this. This is an interesting future direction to explore. Once this is established, we can study the asymptotics of the family of invariant measures of countable-state mean-field models when the limiting dynamics possesses multiple stable equilibria. We can also use this to study the exit time asymptotics, and the large time behaviour and metastability in countable-state mean-field models using ideas similar to those used in Chapter 2.

Generalised quasipotential: The counterexamples in Chapter 4 suggest that there could be a more general notion of the quasipotential that governs the rate function for the family of invariant measures of a broad class of problems. This generalised quasipotential may reduce to the usual Freidlin-Wentzell quasipotential for finite-dimensional problems and the infinite-dimensional problem considered in Chapter 4 with the $1/z$-decay assumption on the transition rates. It would be interesting to look for such a generalised quasipotential.

Models with diminishing transition rates: The main results of this thesis are proved under the assumption that the transition rates of the particles are bounded away from 0. Many models that arise in practice, especially those with a countable state space, have diminishing transition rates as we approach some boundary of the state space; see, for example, Budhiraja et al. [19] for the study of the process-level large deviations of the join-the-shortest queue model. In such situations, the methods studied in this thesis needs to be extended suitably to establish the process-level LDP. It would be interesting to study the large time behaviour and metastability in the join-the-shortest queue model. More generally, it would be interesting to formulate a general mean-field interacting particle system with diminishing transition rates that is applicable for a broad class of problems, study its process-level large deviations, the large time behaviour and the exit time estimates, and the large deviations of the family of invariant measures.

Countable-state models with time scale separation: Another interesting direction is to explore the large deviations of two time scale mean-field models with countable state space, which is an extension to the model studied in Chapter 3. The proof techniques in Chapter 3 relies on the fact that the state space is locally compact, and that nearby points in the space can be connected with trajectories of small cost. These properties are then used to show the necessary regularity properties of the variational problems studied in Chapter 3. If the state space (of the empirical measure process) is not locally compact then the ideas used in the proofs of Chapter 3 are no longer directly applicable. It would be interesting to study this situation.

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